

Cyclic Homology of Strong Smash Product Algebras

Jiao ZHANG and Naihong HU

Abstract

For any strong smash product algebra $A \#_R B$ of two algebras A and B with a bijective morphism R mapping from $B \otimes A$ to $A \otimes B$, we construct a cylindrical module $A \natural B$ whose diagonal cyclic module $\Delta_\bullet(A \natural B)$ is graphically proven to be isomorphic to $C_\bullet(A \#_R B)$ the cyclic module of the algebra. A spectral sequence is established to converge to the cyclic homology of $A \#_R B$. Examples are provided to show how our results work. Particularly, the cyclic homology of the Pareigis' Hopf algebra is obtained in the way.

Keywords: Cyclic homology, strong smash product algebra.

MSC(2000): 19D55, 16S40.

Introduction

Calculating cyclic homology of the crossed product algebra is an attractive problem studied extensively in cyclic homology theory. When G is a discrete group or a compact Lie group and A is an algebra or a C^∞ manifold acted by G , the cyclic homology of the crossed product algebra $A \rtimes G$ is considered by B.L. Feigin and B.L. Tsygan [8], J.L. Brylinski [6], V. Nistor [20], and E. Getzler and J.D.S. Jones [9]. When A is an H -module algebra, where H is a Hopf algebra with an invertible antipode, R. Akbarpour and M. Khalkhali [1] investigated the cyclic homology of the crossed product algebra $A \rtimes H$. Their results generalize the work of Getzler and Jones in [9].

In recent decades there have appeared many kinds of products of different types of algebras in the research of Hopf algebras, for instance, the crossed product, or called (classical) smash product, of a Hopf algebra and its module algebra, Takeuchi's smash product [25] of a left comodule algebra and a left module algebra where the action and the coaction are taken over one Hopf algebra, the tensor product of two algebras in the natural sense or in a braided tensor category, the (generalized) Drinfeld double, double crossproduct of Hopf algebras, etc. These concepts are closely related with the factorization of an algebra into two subalgebras. The algebra factorization is described by S. Majid [19], the generalized factorization problem is stated by S. Caenepeel et al. [7]. A 'generalized braiding', which is quasitriangular and normal, is associated closely with the algebra factorization. When it is a bijection, we call the product algebra a *strong smash product algebra*.

In this paper, we generalize both works of Getzler and Jones [9], and of Akbarpour and Khalkhali [1] to strong smash product algebras. Indeed, the crossed product algebras discussed in [9] and [1] are special examples of the strong smash product algebras. We organize this paper as follows. In Section 1, we give the explicit definition of the strong smash product algebra $A \#_R B$. In Section 2, we construct a cylindrical module $A \natural B$. Using diagrammatical presentations we prove that $\Delta_\bullet(A \natural B)$ the cyclic module related to the diagonal of $A \natural B$ is isomorphic to the cyclic module of $A \#_R B$. In Section 3, we recall some notations and apply the generalized Eilenberg-Zilber theorem for cylindrical modules due to Getzler and Jones [9].

In Section 4, we construct a spectral sequence converging to the cyclic homology of $A\#_R B$. In Section 5, we apply our theorems to Majid's double crossproduct of Hopf algebras after showing that they pertain to the class of strong smash product algebras. As any Drinfeld's quantum double has a double crossproduct structure (see [19]), the notion of strong smash product algebras does cover a wild range of the recent interesting examples, for instance, the two-parameter or multiparameter quantum groups, and the pointed Hopf algebras arising from Nichols algebras of diagonal type (see [2, 3, 4, 5, 10, 12, 13, 14, 22] and references therein). Besides these, another concrete example for the computation of the cyclic homology of the Pareigis' Hopf algebra \mathcal{P} is given to illuminate our results.

We assume that k is a field containing \mathbb{Q} in the whole paper unless otherwise stated. Every algebra in this paper is assumed to be a unital associative k -algebra.

1 Strong smash product algebra

Majid defined in his book [19] an algebra factorization. A unital and associative algebra X *factorizes* through its subalgebras A and B , if the product map defines a linear isomorphism $A \otimes B \cong X$. The necessary and sufficient conditions for the existence of an algebra factorization is the existence of a linear map R from $B \otimes A$ to $A \otimes B$, which is quasitriangular and normal. In [7], the algebra which can be factorized is called a smash product and denoted by $A\#_R B$. In addition, if R is also an isomorphism of vector spaces, we call $A\#_R B$ a *strong* smash product algebra. The explicit definitions are as follows:

Definition 1.1. Let A and B be two algebras, and $R : B \otimes A \rightarrow A \otimes B$ be a linear map. R is called *quasitriangular* if it obeys

$$\begin{aligned} R \circ (m \otimes id) &= (id \otimes m)R_{12}R_{23}, \\ R \circ (id \otimes m) &= (m \otimes id)R_{23}R_{12}, \end{aligned}$$

where m is the product map, $R_{12} = R \otimes id$ and $R_{23} = id \otimes R$.

R is called *normal* if it obeys

$$\begin{aligned} R(1 \otimes a) &= a \otimes 1, \quad \forall a \in A, \\ R(b \otimes 1) &= 1 \otimes b, \quad \forall b \in B. \end{aligned}$$

The *smash product algebra* of A and B with a quasitriangular and normal R , denoted by $A\#_R B$, is defined to be $A \otimes B$ as a vector space equipped with product

$$(a \otimes b)(a' \otimes b') = aR(b \otimes a')b', \quad \forall a, a' \in A, b, b' \in B.$$

The smash product algebra $A\#_R B$ defined above is a unital associative algebra with the unit $1_A \otimes 1_B$. The product of $A\#_R B$ appeared first in [27], where a sufficient condition is given for the product to be associative.

Definition 1.2. The smash product algebra $A\#_R B$ is said to be *strong*, if R is invertible.

Proposition 1.3. $A\#_R B$ is a strong smash product algebra if and only if $B\#_{R^{-1}} A$ is a strong smash product algebra.

Indeed, R is quasitriangular (resp. normal) if and only if R^{-1} is quasitriangular (resp. normal).

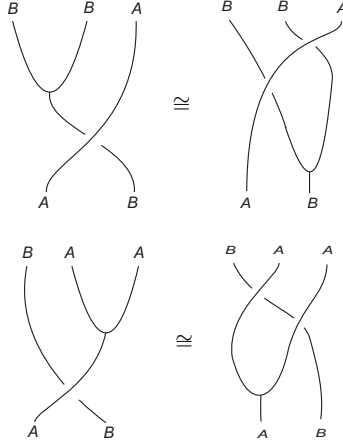
Proof. Since R is invertible with $R^{-1} : A \otimes B \rightarrow B \otimes A$, we have

$$\begin{aligned} R(m_B \otimes id) &= (id \otimes m_B)R_{12}R_{23}, \Leftrightarrow (m_B \otimes id)(R_{12}R_{23})^{-1} = R^{-1}(id \otimes m_B), \\ &\Leftrightarrow R^{-1}(id \otimes m_B) = (m_B \otimes id)R_{23}^{-1}R_{12}^{-1}, \\ R(id \otimes m_A) &= (m_A \otimes id)R_{23}R_{12}, \Leftrightarrow (id \otimes m_A)(R_{23}R_{12})^{-1} = R^{-1}(m_A \otimes id), \\ &\Leftrightarrow R^{-1}(m_A \otimes id) = (id \otimes m_A)R_{12}^{-1}R_{23}^{-1}. \end{aligned}$$

The normalization conditions are clear. \square

It proves very convenient to do computations using diagrammatical presentations. Present the multiplication of an algebra by \bigvee and R from $B \otimes A$ to $A \otimes B$ by $\begin{array}{c} B \quad A \\ \diagdown \quad \diagup \\ A \quad B \end{array}$. Thus R^{-1} can

be presented by $\begin{array}{c} A \quad B \\ \diagup \quad \diagdown \\ B \quad A \end{array}$. The quasitriangular conditions can be diagrammatically expressed as follows:



The concept of smash product algebra $A \#_R B$ recovers the crossed product algebra (or called classical smash product algebra) $A \rtimes H$ and Takeuchi's smash product algebra $A \# B$ (defined in [25]) where H is a Hopf algebra, A is an H -module algebra and B is an H -comodule algebra. The two subalgebras play different roles in $A \rtimes H$ and $A \# B$. One algebra produces action on the other. However, in the strong smash product algebra $A \#_R B$, the status of A and B is equal. They act on each other. The strong smash product algebra $A \#_R B$ is a more natural concept, as in physics the general principle is that every action has a 'reaction'.

Many smash product algebras are strong smash product algebras.

Example 1.4. The tensor product of two algebras in a braided tensor category is a strong smash product algebra. Here R is deduced directly from the braiding in that category, so R is invertible.

Example 1.5. Let H be a Hopf algebra with an invertible antipode S . A is a left H -module algebra and B is a left H -comodule algebra. Takeuchi's smash product $A \# B$ is an algebra with the multiplication $(a \# b)(a' \# b') = a(b_{[-1]} \cdot a') \# b_{[0]} b'$ and the unit $1_A \otimes 1_B$, where $b \mapsto b_{[-1]} \otimes b_{[0]}$ is the left H -comodule structure map for $a, a' \in A$ and $b, b' \in B$. When $B = H$, $A \# B = A \rtimes H$ is the crossed product algebra. Define $R : B \otimes A \rightarrow A \otimes B$ by

$$R(b \otimes a) = b_{[-1]} \cdot a \otimes b_{[0]}.$$

One can check that R is quasitriangular and normal through the definition of the module algebra and the comodule algebra. R has the inverse defined by

$$R^{-1}(a \otimes b) = b_{[0]} \otimes S^{-1}(b_{[-1]}).a,$$

for all $a \in A$ and $b \in B$.

Hence, the crossed product algebras discussed in [9] and [1] are special examples of our strong smash product algebras.

2 Paracyclic modules and cylindrical modules

2.1 From Getzler and Jones' point of view, all the operators of a cyclic module can be generated by only two operators, i.e., the last face map and the extra degeneracy map. Hence we can give an equivalent definition for cyclic modules. In this subsection, k can be a commutative ring.

Definition 2.1. A *cyclic module* is a sequence of k -modules $\{C_n\}_{n \geq 0}$ which is endowed for each n with two k -linear maps $d_n : C_n \rightarrow C_{n-1}$ and $s_{-1} : C_n \rightarrow C_{n+1}$, such that $d_n s_{-1}$ is invertible, and by setting

$$\begin{aligned} (1) \quad & t_n := d_n s_{-1} : C_n \rightarrow C_n, \\ (2) \quad & d_i := t_{n-1}^{-(n-i)} d_n t_n^{n-i} : C_n \rightarrow C_{n-1}, \quad \text{for } 0 \leq i \leq n, \\ & s_i := t_{n+1}^{i+1} s_{-1} t_n^{-(i+1)} : C_n \rightarrow C_{n+1}, \quad \text{for } 0 \leq i \leq n, \end{aligned}$$

for any $i, j \in \mathbb{N}$, the following relations hold

$$\begin{aligned} (3) \quad & d_i d_j = d_{j-1} d_i \quad \text{for } i < j, \\ & s_i s_j = s_{j+1} s_i \quad \text{for } i \leq j, \\ & d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j, \\ id & \text{for } i = j, i = j+1, \\ s_j d_{i-1} & \text{for } i > j+1. \end{cases} \\ (4) \quad & t_n^{n+1} = id. \end{aligned}$$

d_i 's are called *face maps* and s_i 's are called *degeneracy maps* for $i \geq 0$, t is called the *cyclic operator*. s_{-1} is called the *extra degeneracy map* and $d_n : C_n \rightarrow C_{n-1}$ is called the *last face map* for C_n .

Therefore, a cyclic module can be regarded as an underlying simplicial module $\{C_n\}_{n \geq 0}$, whose face maps, degeneracy maps and cyclic operators are generated by the last face map d_n and the extra degeneracy map s_{-1} for each C_n in the way expressed in (1) and (2) satisfying (3) and (4).

If the condition (4) is replaced by

$$(5) \quad d_0 t_n = d_n, \quad s_0 t_n = t_{n+1}^2 s_n,$$

then that sequence of k -modules is called a *paracyclic module*. In fact, the equalities in (5) are consequences of the cyclicity of the invertible operator t , that is, from (4), one can get (5).

For all n , set the following operators

$$\begin{aligned}
(6) \quad & b = \sum_{i=0}^n (-1)^i d_i : C_n \rightarrow C_{n-1}, \\
& T = t^{n+1} : C_n \rightarrow C_n, \\
& N = \sum_{i=0}^n (-1)^{in} t^i : C_n \rightarrow C_n, \\
& B = (1 + (-1)^n t) s_{-1} N : C_n \rightarrow C_{n+1}.
\end{aligned}$$

Lemma 2.2 ([9]). *We have the equalities*

$$bT = Tb, \quad bB + Bb = 1 - T.$$

Getzler and Jones first introduced in [9] the concepts of the bi-paracyclic module and the cylindrical module. We recall their definitions here.

Definition 2.3 ([9],[1]). *A bi-paracyclic module is a sequence of k -modules*

$$(\{C_{m,n}\}_{m,n \geq 0}, d_i^{m,n}, s_i^{m,n}, t_{m,n}, \bar{d}_j^{m,n}, \bar{s}_j^{m,n}, \bar{t}_{m,n})$$

such that $(\{C_{m,n}\}_{m,n \geq 0}, d_i^{m,n}, s_i^{m,n}, t_{m,n})$ and $(\{C_{m,n}\}_{m,n \geq 0}, \bar{d}_j^{m,n}, \bar{s}_j^{m,n}, \bar{t}_{m,n})$ are two paracyclic modules and the operators $d_i^{m,n}, s_i^{m,n}, t_{m,n}$ commute with the operators $\bar{d}_j^{m,n}, \bar{s}_j^{m,n}, \bar{t}_{m,n}$. Moreover, if in addition, $t_{m,n}^{m+1} \bar{t}_{m,n}^{n+1} = id_{m,n}$ for all $m, n \geq 0$, then this bi-paracyclic module is called a *cylindrical module*.

Another interesting concept named parachain complex was also given by Getzler and Jones [9]. The mixed complex defined by Kassel [15] is a special case of parachain complexes. Here we need only the mixed complex. The *mixed complex* is, by definition, a graded k -module $(M_n)_{n \in \mathbb{N}}$ endowed with two graded commutative differentials, one decreasing the degree and the other increasing the degree. That is, (M_\bullet, b, B) with $b : M_n \rightarrow M_{n-1}$ and $B : M_n \rightarrow M_{n+1}$ satisfies $b^2 = B^2 = bB + Bb = 0$. A morphism of mixed complexes (M_\bullet, b, B) to (M'_\bullet, b, B) is a sequence of morphisms $f_k : M_n \rightarrow M'_{n+2k}$ for $k \geq 0$ such that $f = \sum_{k \geq 0} u^k f_k$ commutes with $b + uB$. For a cyclic module (C_\bullet, d_i, s_i, t) associated with the operators b and B defined in (6), (C_\bullet, b, B) is a mixed complex.

It is usually simpler to consider the complex with one differential than to consider the mixed complex with two differentials. Actually, a mixed complex can be converted into a complex. Let V_\bullet be a non-negative graded k -module. Denote by $V_\bullet[[u]]$ the graded k -modules of formal power series in a variable u with coefficients in V_\bullet . Set the degree of u be -2 . If V_\bullet is endowed with a degree -1 endomorphism b and a degree 1 endomorphism B , then (V_\bullet, b, B) is a mixed complex if and only if $(V_\bullet[[u]], b + uB)$ is a complex with the differential $b + uB$. Here set $V_n[[u]] = \sum_{i \geq 0} V_{n+2i} u^i$.

2.2 Now we return to our strong smash product algebra.

The cyclic module $C_\bullet(A \#_R B)$ of an algebra $A \#_R B$ is defined as usual (see [17] etc). That is, $C_n(A \#_R B) = (A \#_R B)^{\otimes(n+1)}$ for all $n \in \mathbb{N}$ with

$$\begin{aligned}
d_i(x_0, \dots, x_n) &= (x_0, \dots, x_i x_{i+1}, \dots, x_n), \quad 0 \leq i < n, \\
d_n(x_0, \dots, x_n) &= (x_n x_0, \dots, x_{n-1}), \\
t(x_0, \dots, x_n) &= (x_n, x_0, \dots, x_{n-1}),
\end{aligned}$$

$$s_j(x_0, \dots, x_n) = (x_0, \dots, x_j, 1, x_{j+1}, \dots, x_n), \quad 0 \leq j \leq n,$$

where $x_0, \dots, x_n \in A \#_R B$.

For A and B the subalgebras of $A \#_R B$, we introduce a cylindrical module denoted by $A \natural B$ which generalizes the cylindrical module constructed in the paper [9] by Getzler and Jones where B is a group algebra and A is a B -module algebra, also generalizes the cylindrical module constructed in the paper [1] by Akbarpour and Khalkhali where B is a Hopf algebra with an invertible antipode and A is a B -module algebra.

For $p, q \in \mathbb{N}$, set $A \natural B(p, q) = B^{\otimes(p+1)} \otimes A^{\otimes(q+1)}$ endowed with the following operators which are mainly defined on B 's side:

$$(7) \quad \begin{aligned} t_{p,q}(b_0, \dots, b_p | a_0, \dots, a_q) &= f^{p+q+1,1}(b_0, \dots, b_{p-1}, \Theta_q(b_p, a_0, \dots, a_q)), \\ d_i^{p,q}(b_0, \dots, b_p | a_0, \dots, a_q) &= (b_0, \dots, b_i b_{i+1}, \dots, b_p | a_0, \dots, a_q), \quad 0 \leq i < p, \\ s_i^{p,q}(b_0, \dots, b_p | a_0, \dots, a_q) &= (b_0, \dots, b_i, 1, b_{i+1}, \dots, b_p | a_0, \dots, a_q), \quad 0 \leq i \leq p; \end{aligned}$$

and the following operators which are mainly defined on A 's side:

$$(8) \quad \begin{aligned} \bar{t}_{p,q}(b_0, \dots, b_p | a_0, \dots, a_q) &= (\Gamma_p(a_q, b_0, \dots, b_p), a_0, \dots, a_{q-1}), \\ \bar{d}_j^{p,q}(b_0, \dots, b_p | a_0, \dots, a_q) &= (b_0, \dots, b_p | a_0, \dots, a_j a_{j+1}, \dots, a_q), \quad 0 \leq j < q, \\ \bar{s}_j^{p,q}(b_0, \dots, b_p | a_0, \dots, a_q) &= (b_0, \dots, b_p | a_0, \dots, a_j, 1, a_{j+1}, \dots, a_q), \quad 0 \leq j \leq q, \end{aligned}$$

where $f^{\cdot, \cdot}$ is the flip map defined by

$$f^{m,n}(c_1, \dots, c_m, c'_1, \dots, c'_n) := (c'_1, \dots, c'_n, c_1, \dots, c_m),$$

Θ_q is a composition of R 's defined by

$$\Theta_q := R_{q+1,q+2} \cdots R_{23} R_{12} : B \otimes A^{\otimes(q+1)} \rightarrow A^{\otimes(q+1)} \otimes B,$$

and Γ_p is a composition of R^{-1} 's defined by

$$\Gamma_p := R_{p+1,p+2}^{-1} \cdots R_{23}^{-1} R_{12}^{-1} : A \otimes B^{\otimes(p+1)} \rightarrow B^{\otimes(p+1)} \otimes A,$$

for $a_i \in A$ and $b_i \in B$. Define the last face maps by $d_p^{p,q} = d_0^{p,q} t_{p,q}$ and $\bar{d}_q^{p,q} = \bar{d}_0^{p,q} \bar{t}_{p,q}$.

We can simply write $t_{p,q} = f^{p+q+1,1} \circ (id^{\otimes p} \otimes \Theta_q)$ and $\bar{t}_{p,q} = (\Gamma_p \otimes id^{\otimes q}) \circ f^{p+q+1,1}$. Graphically, present the flip map between $A \otimes B$ and $B \otimes A$ by $\begin{array}{c} B \ A \\ \times \\ A \ B \end{array}$, its inverse is $\begin{array}{c} A \ B \\ \times \\ B \ A \end{array}$. The

identity is denoted by $\begin{array}{c} | \\ | \\ | \end{array}$. Then $t_{p,q}$ and $\bar{t}_{p,q}$ can be presented by

$$t_{p,q} = \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & \dots & p-1 & p & 0 & 1 & \dots & q \end{array} \\ \begin{array}{c} \text{Diagram of } t_{p,q} \text{ showing thick lines for } A \text{ and thin lines for } B. \end{array} \end{array}, \quad \bar{t}_{p,q} = \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & \dots & p & 0 & 1 & \dots & q-1 & q \end{array} \\ \begin{array}{c} \text{Diagram of } \bar{t}_{p,q} \text{ showing thick lines for } A \text{ and thin lines for } B. \end{array} \end{array}.$$

The elements in A are drawn with thick lines and the elements in B are drawn with thin lines in order to show differences.

Since $RR^{-1} = R^{-1}R = id$, we have

(I)

Although R does not satisfy the braid relations, the flip maps always satisfy them and are involutions. When the three crosses in one side of the braid relations consist of two flip maps and one R or R^{-1} , we still have the “braid” relations.

Lemma 2.4. $f_{12}f_{23}R_{12} = R_{23}f_{12}f_{23}$, $f_{12}R_{23}f_{12} = f_{23}R_{12}f_{23}$, $R_{12}f_{23}f_{12} = f_{23}f_{12}R_{23}$, where f denotes $f^{1,1}$, i.e., the flip map of two elements. The graphical notations are

$$(II)$$

For R^{-1} , we have the same relations.

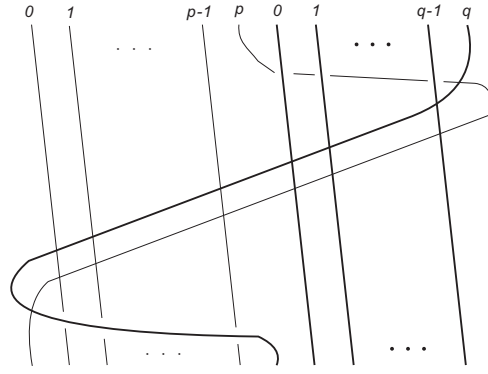
Proposition 2.5. $(A \sharp B, d_i, s_i, t, \bar{d}_j, \bar{s}_j, \bar{t})$ is a cylindrical module.

Proof. We check the commutativity of the barred operators and unbarred operators first. We would like to use the graphical proof.

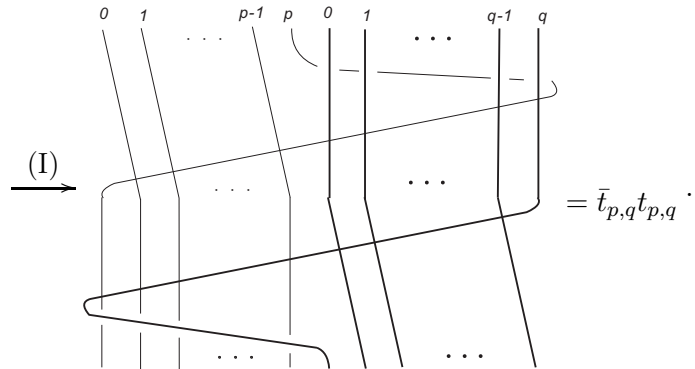
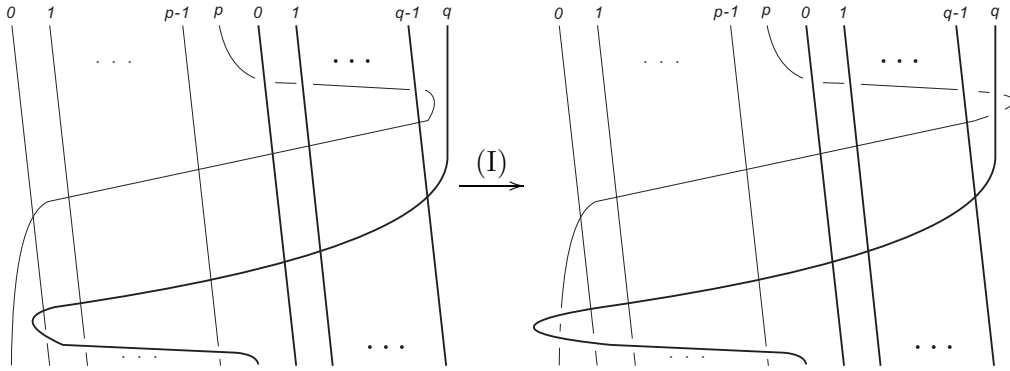
(i) : $t_{p,q} \bar{t}_{p,q} =$

act (I)

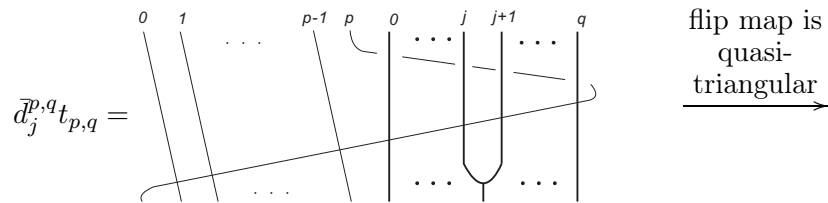
act (II) on
its crosses in
the upper right corner
and the
lower left corner



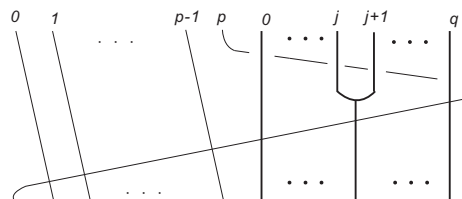
the flip maps
are involutions
and obey
braid relations



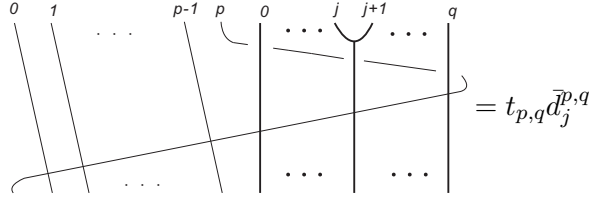
(ii) For $0 \leq j < q$,



flip map is
quasi-
triangular



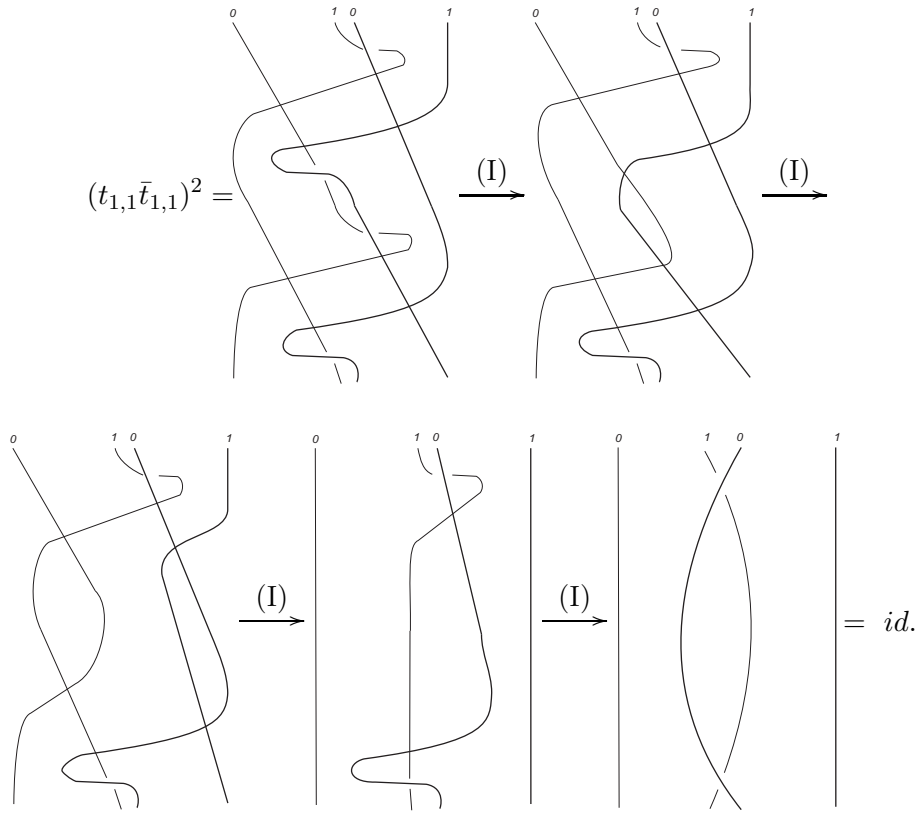
R is quasi-
triangular



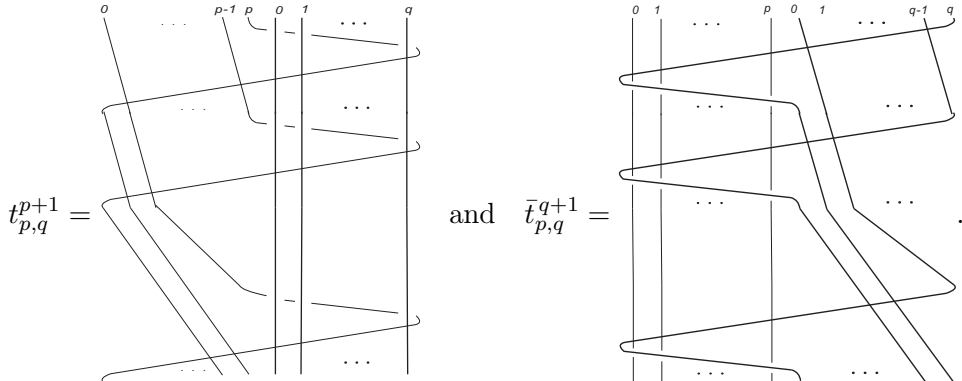
Similar proof holds for $d_i^{p,q} \bar{t}_{p,q} = \bar{t}_{p,q} d_i^{p,q}$, for $0 \leq i < p$.

The flip map and $R^{\pm 1}$ are quasitriangular and normal, $d_p^{p,q} = d_0^{p,q} t_{p,q}$ and $\bar{d}_q^{p,q} = \bar{d}_0^{p,q} \bar{t}_{p,q}$, so the other commutative equalities can be proved easily.

For the cylindrical condition, we use inductions on p and q . For $p = q = 1$, using the fourth picture in the process of turning $t_{p,q} \bar{t}_{p,q}$ to $\bar{t}_{p,q} t_{p,q}$, we get

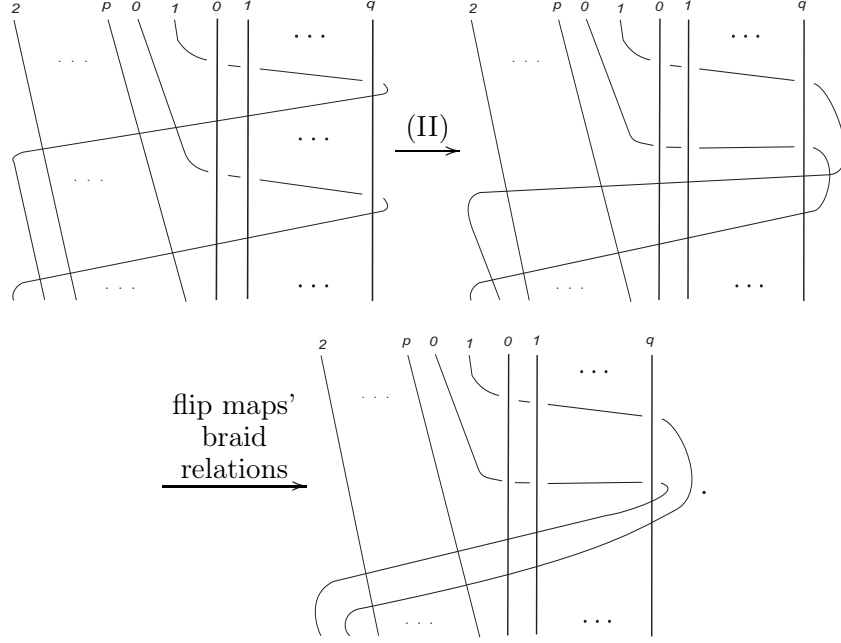


Suppose that $t_{m,n}^{m+1} \bar{t}_{m,n}^{n+1} = id_{m,n}$ for $\forall m < p$ and $\forall n < q$, we need to prove $t_{p,q}^{p+1} \bar{t}_{p,q}^{q+1} = id_{p,q}$. We have



We can use bands in graphs to stand for parallel lines, that is, lines without any intersections or crosses between themselves.

If we can draw the elements b_0, \dots, b_{p-1} of B together by a grey band, and draw the elements a_0, \dots, a_{q-1} of A together by a black band, then using the movements for the case $m = n = 1$, we will get the proposition. We just give the equivalent moves for turning the lines b_0 and b_1 in the graph of $t_{p,q}^{p+1}$ to parallel lines, others can be done by similar moves. The only intersections between b_0 and b_1 occur while doing the p -th and $p+1$ -th powers of $t_{p,q}$. So we concentrate on that part of graph.



□

Let $\Delta_\bullet(A\sharp B)$ be the *diagonal* of the cylindrical module $A\sharp B$, i.e.,

$$\Delta_n(A\sharp B) = A\sharp B(n, n).$$

It is a cyclic module with face maps $d_i = d_i^{n,n} \bar{d}_i^{n,n}$, degeneracy maps $s_i = s_i^{n,n} \bar{s}_i^{n,n}$ and the cyclic operator $t_n = t_{n,n} \bar{t}_{n,n}$.

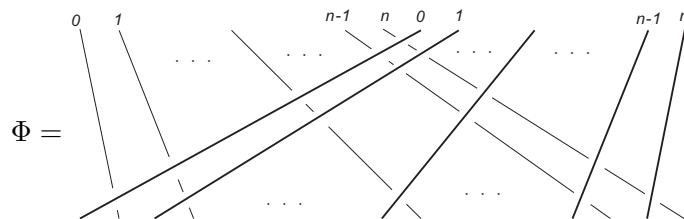
Proposition 2.6. $\Delta_\bullet(A\sharp B)$ is isomorphic to $C_\bullet(A\#_R B)$ as cyclic modules.

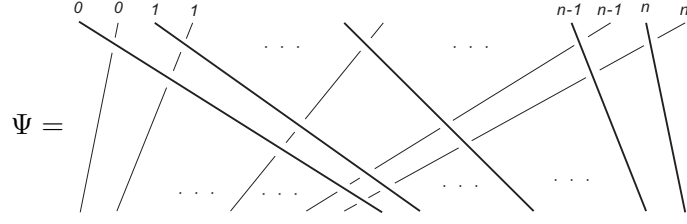
Proof. Define morphisms $\Phi_n : A\sharp B(n, n) \rightarrow C_n(A\#_R B)$ by

$$\Phi = R_{2n+1, 2n+2} (R_{2n-1, 2n} R_{2n, 2n+1}) \cdots (R_{12} R_{23} \cdots R_{n+1, n+2}),$$

and $\Psi_n : C_n(A\#_R B) \rightarrow A\sharp B(n, n)$ by

$$\Psi = R_{n+1, n+2}^{-1} (R_{n, n+1}^{-1} R_{n+2, n+3}^{-1}) \cdots (R_{12}^{-1} R_{34}^{-1} \cdots R_{2n+1, 2n+2}^{-1}).$$

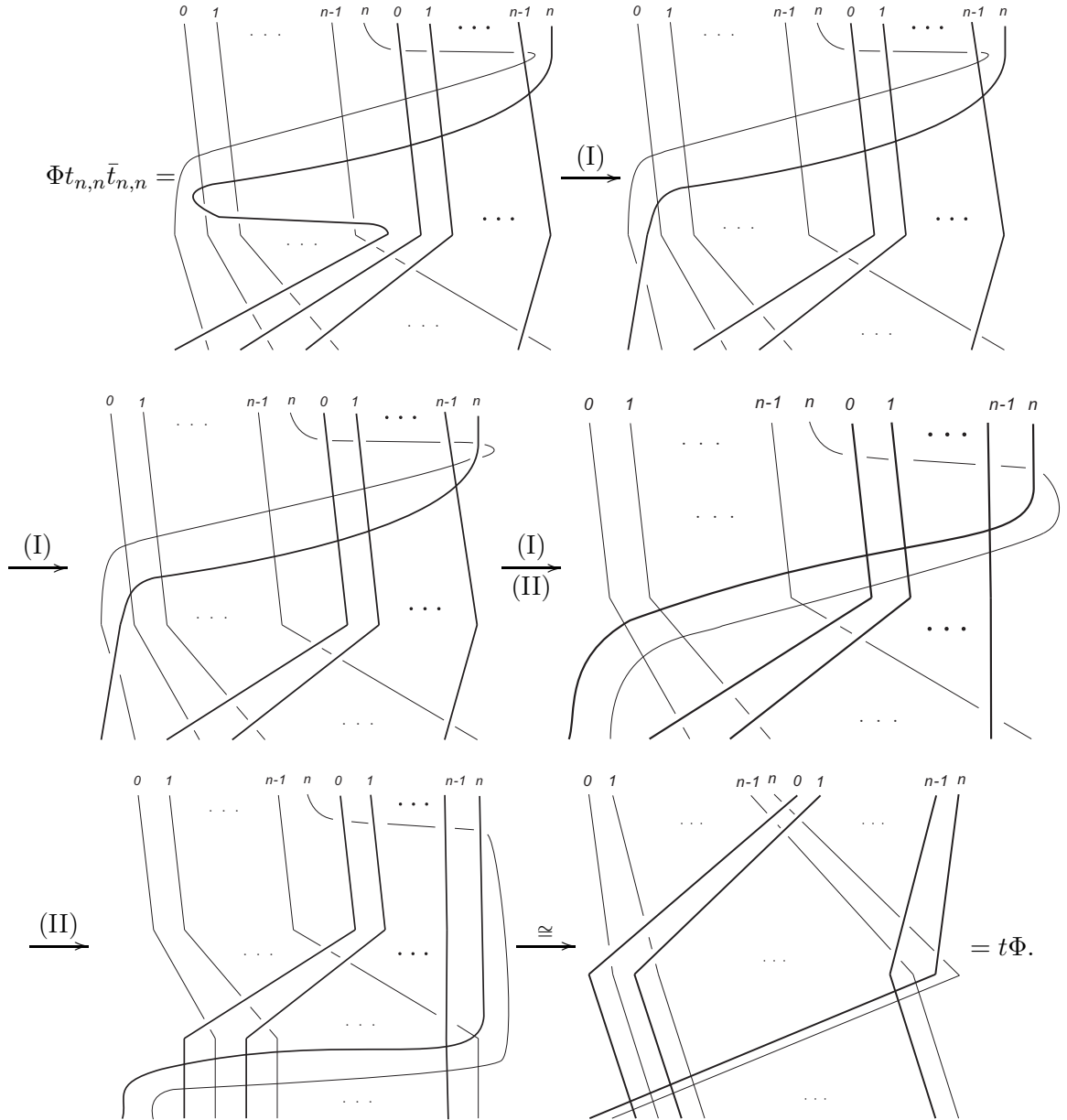




Note that $R_{i,i+1}^{\pm 1} R_{j,j+1}^{\pm 1} = R_{j,j+1}^{\pm 1} R_{i,i+1}^{\pm 1}$ for $|i - j| > 1$. So Φ and Ψ are inverses to each other.

We need to prove that Φ and Ψ are morphisms of cyclic modules. We only show that Φ commutes with the cyclic operator and the face maps. It is similar for Ψ .

Again using the fourth picture in the process of turning $t_{p,q} \bar{t}_{p,q}$ to $\bar{t}_{p,q} t_{p,q}$, we get



Since R and R^{-1} are quasitriangular, for $0 \leq i < n$,

$$d_i \Phi = \text{[Diagram 1]} \cong \text{[Diagram 2]} = \Phi \bar{d}_i^{n,n} d_i^{n,n}.$$

The diagram shows two complex structures of lines and crossings. The first diagram, labeled $d_i \Phi$, has lines labeled with indices $0, 1, \dots, i, i+1, \dots, n-1, n, 0, 1, \dots, i, i+1, \dots, n-1, n$. It features a crossing between lines i and $i+1$. The second diagram, labeled \cong , is similar but with a different crossing configuration. The third part of the equation shows the result as $\Phi \bar{d}_i^{n,n} d_i^{n,n}$.

□

3 Application of the generalized Eilenberg-Zilber theorem

Let $(\{C_{m,n}\}_{m,n \geq 0}, d_i^{m,n}, s_i^{m,n}, t_{m,n}, \bar{d}_j^{m,n}, \bar{s}_j^{m,n}, \bar{t}_{m,n})$ be a cylindrical module. We can set as in (6) the degree -1 endomorphism b (resp. \bar{b}), the degree 1 endomorphism B (resp. \bar{B}) and the degree 0 endomorphism T (resp. \bar{T}) associated with d_i, s_i, t (resp. $\bar{d}_j, \bar{s}_j, \bar{t}$). The total paracyclic complex is a mixed complex. Explicitly, let $\mathbb{C}_n = \bigoplus_{i+j=n} C_{i,j}$, $\mathbb{b} = b + \bar{b}$ and $\mathbb{B} = B + T \bar{B}$. Since $C_{\bullet, \bullet}$ is a cylindrical module, $T\bar{T} = 1$, $\bar{b}B = -B\bar{b}$ and $\bar{B}b = -b\bar{B}$. Then by Lemma 2.2,

$$\begin{aligned} \mathbb{b}\mathbb{B} + \mathbb{B}\mathbb{b} &= (b + \bar{b})(B + T \bar{B}) + (B + T \bar{B})(b + \bar{b}) \\ &= bB + Bb + \bar{b}B + B\bar{b} + T(\bar{B}b + b\bar{B}) + T(\bar{b}\bar{B} + \bar{B}\bar{b}) \\ &= 1 - T + T(1 - \bar{T}) = 0, \end{aligned}$$

$(\mathbb{C}_{\bullet}, \mathbb{b}, \mathbb{B})$ is a mixed complex.

The generalized Eilenberg-Zilber theorem for paracyclic modules was proved by Getzler and Jones [9] using topological method, later it was reproved by Khalkhali and Rangipour [16] using an algebraic method. The theorem tells us that, for a cylindrical module there exists a quasi-isomorphism from its total mixed complex to its diagonal mixed complex. Due to Proposition 2.6, we have:

Theorem 3.1. *Let $A \#_R B$ be a strong smash product algebra, $A \sharp B$ a cylindrical module defined in (7) and (8). Then there exists a quasi-isomorphism of mixed complexes $\text{Tot}_{\bullet}(A \sharp B)$ and $C_{\bullet}(A \#_R B)$.*

It was discovered by Getzler and Jones [9] that the Hochschild homology, the cyclic homology, the negative cyclic homology and the periodic cyclic homology can be unified to be cyclic homologies of a mixed complex with coefficients. Specifically, let M_{\bullet} be a mixed complex and W be a graded $k[u]$ -module, denote $M_{\bullet}[[u]] \otimes_{k[u]} W$ by $M_{\bullet} \boxtimes W$. Note that this tensor product is a graded tensor product. Let (C_{\bullet}, b, B) be the mixed complex associated to its cyclic module structure. $C_{\bullet} \boxtimes W$ is a complex with the differential $(b + uB) \otimes_{k[u]} id_W$. Call the homology of the complex $C_{\bullet} \boxtimes W$ the cyclic homology of the mixed complex of C_{\bullet} with coefficients in W and denote it by $HC_*(C_{\bullet}; W)$. Then for $W = k[u]$ (resp. $k[u, u^{-1}]$, $k[u, u^{-1}]/uk[u]$ and $k[u]/uk[u]$) $HC_*(C_{\bullet}; W) = HC_*(C_{\bullet})$ (resp. $HP_*(C_{\bullet})$, $HC_*(C_{\bullet})$ and $HH_*(C_{\bullet})$). If C is the usual cyclic module associated with an algebra A , then we simply denote $HC_*(C_{\bullet}(A); W)$ by $HC_*(A; W)$.

The first author would like to thank Professor Getzler for pointing out the flatness condition concealed here, which turns out useful in the consequent arguments.

Lemma 3.2. *Let k be a field, V a k -vector space and u a variable. Then $V[[u]]$ is a flat $k[u]$ -module.*

Proof. Since $k[u]$ is a principal ideal domain, a $k[u]$ -module is flat if and only if it is torsion-free. Clearly, $V[[u]]$ is a torsion-free $k[u]$ -module. \square

Lemma 3.3. *Let \mathcal{R} be a ring, M a left \mathcal{R} -module and P_\bullet, Q_\bullet bounded below complexes of flat right \mathcal{R} -modules. If P_\bullet and Q_\bullet are quasi-isomorphic, then*

$$H_n(P_\bullet \otimes_{\mathcal{R}} M) \cong H_n(Q_\bullet \otimes_{\mathcal{R}} M).$$

Proof. We know from [28] that, for bounded below complexes of flat right \mathcal{R} -modules P_\bullet and Q_\bullet ,

$$H_n(P_\bullet \otimes_{\mathcal{R}} M) \cong \text{Tor}_n^{\mathcal{R}}(P_\bullet, M) \text{ and } H_n(Q_\bullet \otimes_{\mathcal{R}} M) \cong \text{Tor}_n^{\mathcal{R}}(Q_\bullet, M)$$

for each n , where Tor is the hypertor. And we have spectral sequences converging to them, that is,

$$\begin{aligned} E_{p,q}^2(P) &= \text{Tor}_p^{\mathcal{R}}(H_q(P_\bullet), M) \Rightarrow \text{Tor}_{p+q}^{\mathcal{R}}(P_\bullet, M), \\ E_{p,q}^2(Q) &= \text{Tor}_p^{\mathcal{R}}(H_q(Q_\bullet), M) \Rightarrow \text{Tor}_{p+q}^{\mathcal{R}}(Q_\bullet, M). \end{aligned}$$

$E_{p,q}^2(P) \cong E_{p,q}^2(Q)$ for all p, q , as $H_q(P_\bullet) \cong H_q(Q_\bullet)$. It yields that $\text{Tor}_n^{\mathcal{R}}(P_\bullet, M) \cong \text{Tor}_n^{\mathcal{R}}(Q_\bullet, M)$ by using the mapping lemma for E^∞ (see [28]). \square

The above two lemmas still hold in the graded module category.

Corollary 3.4 ([9]). *If there exists $f : C \rightarrow C'$ is a quasi-isomorphism of mixed complexes, then for any graded $k[u]$ -module W , we have an isomorphism of cyclic homology groups*

$$\text{HC}_\bullet(C; W) \cong \text{HC}_\bullet(C'; W).$$

Using the generalized Eilenberg-Zilber theorem for paracyclic modules, we have

Corollary 3.5. *Let $A \#_R B$ be a strong smash product algebra, $A \natural B$ be the cylindrical module defined in (7) and (8). Then*

$$\text{HC}_*(A \#_R B; W) \cong \text{HC}_*(\Delta(A \natural B); W) \cong \text{HC}_*(\text{Tot}(A \natural B); W).$$

The following corollary will be used in the next section.

Corollary 3.6. *Let $(\mathfrak{C}, \mathfrak{d})$ be a complex of k -modules and W a graded $k[u]$ -module. Then for each n ,*

$$H_n(\mathfrak{C}[[u]] \otimes_{k[u]} W) = H_n(\mathfrak{C})[[u]] \otimes_{k[u]} W.$$

Proof. Since $\mathfrak{C}[[u]]$ is a complex of flat $k[u]$ -modules,

$$H_n(\mathfrak{C}[[u]] \otimes_{k[u]} W) = \text{Tor}_n^{k[u]}(\mathfrak{C}[[u]], W).$$

Note that the differential of the complex $\mathfrak{C}[[u]]$ does not depend on u . We have a spectral sequence converging to the hypertor whose E^2 -term is

$$\text{Tor}_p^{k[u]}(H_q(\mathfrak{C})[[u]], W).$$

Because $H_q(\mathfrak{C})[[u]]$ is also a flat $k[u]$ -module, the spectral sequence collapses. We get

$$H_n(\mathfrak{C}[[u]] \otimes_{k[u]} W) = \text{Tor}_n^{k[u]}(\mathfrak{C}[[u]], W) = H_n(\mathfrak{C})[[u]] \otimes_{k[u]} W.$$

This completes the proof. \square

4 Cyclic homology of a strong smash product algebra

We can also construct a spectral sequence to calculate the cyclic homology of a strong smash product algebra $A \#_R B$. This is the same as calculating the cyclic homology of $\text{Tot}(A \natural B)$. The first column of the cylindrical module $A \natural B$ plays an important role. Denote by $C_\bullet(A \natural B)$ this paracyclic module $A \natural B(\bullet, 0)$.

Lemma 4.1. *For each $n \in \mathbb{N}$, $C_n(A \natural B)$ is an A -bimodule via the left A -module action*

$$a.(b_0, \dots, b_n | a_0) = (id^{\otimes(n+1)} \otimes m_A)(\Gamma_n(a, b_0, \dots, b_n) | a_0)$$

and the right A -module action

$$(b_0, \dots, b_n | a_0).a = (b_0, \dots, b_n | a_0 a)$$

where Γ_n is defined in Section 2, and $a_0, a \in A, b_j \in B$.

Proof. The right action is trivial. By Proposition 1.3, R^{-1} is also quasitriangular and normal, then the left A -module action is well-defined. And both actions are compatible. \square

For each $p \in \mathbb{N}$, we can define a *Hochschild complex* $(C_\bullet(A, C_p(A \natural B)), d)$, whose homology is the *Hochschild homology of the algebra A with coefficients in $C_p(A \natural B)$* (see [17]). The Hochschild complex is defined explicitly as follows: for any $q \in \mathbb{N}$,

$$C_q(A, C_p(A \natural B)) = C_p(A \natural B) \otimes A^{\otimes q} = B^{\otimes(p+1)} \otimes A \otimes A^{\otimes q},$$

the differential $d : C_q(A, C_p(A \natural B)) \rightarrow C_{q-1}(A, C_p(A \natural B))$ is

$$\begin{aligned} d(b_0, \dots, b_p | a_0 | a_1, \dots, a_q) &= ((b_0, \dots, b_p | a_0).a_1 | a_2, \dots, a_q) \\ &+ \sum_{i=1}^{q-1} (-1)^i (b_0, \dots, b_p | a_0 | a_1, \dots, a_i a_{i+1}, \dots, a_q) \\ &+ (-1)^q (a_q.(b_0, \dots, b_p | a_0) | a_1, \dots, a_{q-1}). \end{aligned} \tag{9}$$

Denote this Hochschild homology by $H_\bullet(A, C_p(A \natural B))$.

Corollary 4.2. *$C_\bullet(A, C_\bullet(A \natural B))$ is a cylindrical module with the same operators defined for $A \natural B$ in (7) and (8).*

Indeed, for each $p, q \in \mathbb{N}$,

$$C_q(A, C_p(A \natural B)) = C_p(A \natural B) \otimes A^{\otimes q} = B^{\otimes(p+1)} \otimes A \otimes A^{\otimes q} = A \natural B(p, q).$$

Note that \bar{b} of $A \natural B$ is exactly the differential d defined in (9).

Define $C_\bullet^A(A \natural B)$ as the *co-invariant space* of $C_\bullet(A \natural B)$ under the left and right actions of A constructed in Lemma 4.1, i.e.,

$$C_\bullet^A(A \natural B) = C_\bullet(A \natural B) / \text{span}\{a.x - x.a \mid a \in A, x \in C_\bullet(A \natural B)\}.$$

And we define the following operators on $C_{\bullet}^A({}_A^{\natural}B)$:

$$(10) \quad \begin{aligned} \tau_n(b_0, \dots, b_n | a) &= f^{n+1,1}(b_0, \dots, b_{n-1}, R(b_n \otimes a)), \\ \partial_i(b_0, \dots, b_n | a) &= (b_0, \dots, b_i b_{i+1}, \dots, b_n | a), \quad \text{for } 0 \leq i < n, \\ \partial_n(b_0, \dots, b_n | a) &= \partial_0 \tau_n(b_0, \dots, b_n | a), \\ \sigma_j(b_0, \dots, b_n | a) &= (b_0, \dots, b_j, 1, \dots, b_n | a), \quad \text{for } 0 \leq j \leq n. \end{aligned}$$

Use the following notations as in [7], for R ,

$$R(b \otimes a) = a^R \otimes b^R, \quad R_{23}R_{12}(b \otimes a_1 \otimes a_2) = a_1^{R_1} \otimes a_2^{R_2} \otimes b^{R_1 R_2}, \text{ etc,}$$

and for R^{-1} ,

$$R^{-1}(a \otimes b) = b^r \otimes a^r, \quad R_{23}^{-1}R_{12}^{-1}(a \otimes b_1 \otimes b_2) = b_1^{r_1} \otimes b_2^{r_2} \otimes a^{r_1 r_2}, \text{ etc,}$$

where $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Then one can check that these operators in (10) are well defined on the co-invariant space. For example,

$$\begin{aligned} \tau(a.(b_0, \dots, b_n | a_0)) &= \tau(b_0^{r_1}, b_1^{r_2}, \dots, b_n^{r_{n+1}} | a^{r_1 \cdot \dots \cdot r_{n+1}} a_0) \\ &= (b_n^{r_{n+1} R}, b_0^{r_1}, b_1^{r_2}, \dots, b_{n-1}^{r_n} | (a^{r_1 \cdot \dots \cdot r_{n+1}} a_0)^R) \\ &= (b_n^{r_{n+1} R_1 R_2}, b_0^{r_1}, b_1^{r_2}, \dots, b_{n-1}^{r_n} | a^{r_1 \cdot \dots \cdot r_{n+1} R_1} a_0^{R_2}) \\ &= (b_n^R, b_0^{r_1}, b_1^{r_2}, \dots, b_{n-1}^{r_n} | a^{r_1 \cdot \dots \cdot r_n} a_0^R) \\ &= a^{R_2} . (b^{R_1 R_2}, b_0, \dots, b_{n-1} | a_0^{R_1}) \quad , \end{aligned}$$

$$\begin{aligned} \tau((b_0, \dots, b_n | a_0).a) &= \tau(b_0, \dots, b_n | a_0 a) \\ &= (b_n^R, b_0, \dots, b_{n-1} | (a_0 a)^R) \\ &= (b_n^{R_1 R_2}, b_0, \dots, b_{n-1} | a_0^{R_1} a^{R_2}) \\ &= (b_n^{R_1 R_2}, b_0, \dots, b_{n-1} | a_0^{R_1}). a^{R_2} \quad . \end{aligned}$$

Proposition 4.3. $C_{\bullet}^A({}_A^{\natural}B)$ is a cyclic module with operators defined in (10).

Proof. We only check that $\tau_n^{n+1} = id$. The other identities are similar to check. In the coinvariant subspace, we have

$$\begin{aligned} \tau^{n+1}(b_0, \dots, b_n | a) &= \tau^n(b_n^R, b_0, \dots, b_{n-1} | a^R) \\ &= \tau^{n-1}(b_{n-1}^{R_2}, b_n^{R_1}, b_0, \dots, b_{n-2} | a^{R_1 R_2}) = \dots \\ &= (b_0^{R_{n+1}}, \dots, b_{n-1}^{R_2}, b_n^{R_1} | a^{R_1 R_2^{\cdot \cdot \cdot R_{n+1}}}) \\ &= (b_0^{R_{n+1}}, \dots, b_{n-1}^{R_2}, b_n^{R_1} | 1). a^{R_1 R_2^{\cdot \cdot \cdot R_{n+1}}} \\ &= a^{R_1 R_2^{\cdot \cdot \cdot R_{n+1}}} . (b_0^{R_{n+1}}, \dots, b_{n-1}^{R_2}, b_n^{R_1} | 1) \\ &= (b_0, \dots, b_n | a). \end{aligned}$$

□

In fact, the above proposition is a special case of the following theorem.

Theorem 4.4. *For any $q \in \mathbb{N}$, $H_q(A, C_\bullet(\mathbb{A}_A B))$ is a cyclic module with (d_i, s_j, t) induced from operators of $A\sharp B$ defined in (7). Especially, we have*

$$H_0(A, C_\bullet(\mathbb{A}_A B)) = C_\bullet^A(\mathbb{A}_A B).$$

Proof. We need to check that, $t_{n,q}^{n+1}$ inducing on $H_q(A, C_n(\mathbb{A}_A B))$ turns out to be identity. For any $x \in H_q(A, C_n(\mathbb{A}_A B))$, $d(x) = 0$, or equivalently, $\bar{b}(x) = 0$,

$$\begin{aligned} (t_{n,q}^{n+1} - id)(x) &= (t_{n,q}^{n+1} - t_{n,q}^{n+1} \bar{t}_{n,q}^{q+1})(x) = t_{n,q}^{n+1} (1 - \bar{t}_{n,q}^{q+1})(x) \\ &= t_{n,q}^{n+1} (\bar{b} \bar{B} + \bar{B} \bar{b})(x) = \bar{b} \bar{B} t_{n,q}^{n+1}(x) = 0 \in H_q(A, C_n(\mathbb{A}_A B)). \end{aligned}$$

Since the barred operators commute with the unbarred operators, all unbarred operators (d_i, s_j, t) are well-defined on $H_q(A, C_\bullet(\mathbb{A}_A B))$ preserving the relations (3) and (5). \square

Lemma 4.5. *The homology group of the complex*

$$\cdots \rightarrow C_q(A, C_\bullet(\mathbb{A}_A B))[[u]] \xrightarrow{d} C_{q-1}(A, C_\bullet(\mathbb{A}_A B))[[u]] \rightarrow \cdots$$

is $H_q(A, C_\bullet(\mathbb{A}_A B))[[u]]$, for each q .

By Corollary 3.5, in order to calculate the cyclic homology of the strong smash algebra $A\#_R B$ with coefficients in W , we can compute the cyclic homology of $\text{Tot}(A\sharp B)$ with coefficients in W , that is, the homology of the complex

$$(\text{Tot}(A\sharp B) \boxtimes W, (b + \bar{b} + uB + uT\bar{B}) \otimes id).$$

We define a filtration on $\text{Tot}(A\sharp B) \boxtimes W$ by rows. Set

$$F_n^p(\text{Tot}(A\sharp B) \boxtimes W) = \sum_{\substack{i+j=n+2l, \\ i \leq p+2l, \\ l \geq 0}} (B^{\otimes(i+1)} \otimes A^{\otimes(j+1)}) u^l \otimes_{k[u]} W, \text{ for } p \geq 0;$$

and $F_n^p(\text{Tot}(A\sharp B) \boxtimes W) = 0$, for $p < 0$.

The spectral sequence $E_{p,q}^r$ of this filtration with $d^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ starts from

$$E_{p,q}^0 = \sum_{l \geq 0} (B^{\otimes(p+2l+1)} \otimes A^{\otimes(q+1)}) u^l \otimes_{k[u]} W,$$

equipped with $d^0 = \bar{b} \otimes id : E_{p,q}^0 \rightarrow E_{p,q-1}^0$.

Recall that $C_q(A, C_\bullet(\mathbb{A}_A B))[[u]] = \sum_{p,l \geq 0} (B^{\otimes(p+2l+1)} \otimes A^{\otimes(q+1)}) u^l$,

$$E_{\bullet,q}^0 = C_q(A, C_\bullet(\mathbb{A}_A B)) \boxtimes W.$$

So from Lemma 4.5 and Corollary 3.6, we get:

Lemma 4.6. *The E^1 -term of the spectral sequence is*

$$E_{\bullet,q}^1 = H_q(A, C_\bullet(\mathbb{A}_A B)) \boxtimes W,$$

equipped with $d^1 : E_{p,q}^1 \rightarrow E_{p-1,q}^1$ that is induced by $(b + uB) \otimes id$.

Theorem 4.7. *The E^2 -term of the spectral sequence is identified with the cyclic homology of the cyclic module $H_\bullet(A, C_\bullet(A^{\natural}_B))$ with coefficients in W . It converges to the cyclic homology of the strong smash product algebra $A\#_R B$ with coefficients in W . That is,*

$$E_{p,q}^2 = \mathrm{HC}_p\left(H_q(A, C_\bullet(A^{\natural}_B)); W\right) \Rightarrow \mathrm{HC}_{p+q}(A\#_R B; W).$$

In parallel, one can also consider the bottom row of the cylindrical module $A\natural B$. We just state the process and indicate the differences here. We skip proofs which are similar as in previous discussions.

Denote the paracyclic module $A\natural B(0, \bullet)$ by $C_\bullet(A^{\natural}_B)$.

Lemma 4.8. *For each $n \in \mathbb{N}$, $C_n(A^{\natural}_B)$ is a B -bimodule via the left B -module action*

$$b.(b_0 | a_0, \dots, a_n) = (bb_0 | a_0, \dots, a_n)$$

and the right B -module action

$$(b_0 | a_0, \dots, a_n).b = (m_B \otimes id^{\otimes(n+1)})(b_0 | \Theta_n^{-1}(a_0, \dots, a_n, b)),$$

where Θ_n is defined in Section 2, and $a_i \in A, b_0, b \in B$.

For each $q \in \mathbb{N}$, a Hochschild complex $(C_\bullet(B, C_q(A^{\natural}_B)), \delta)$ can be defined, its homology is the Hochschild homology of the algebra B with coefficients in $C_q(A^{\natural}_B)$. The Hochschild complex is defined explicitly as follows: For any $p \in \mathbb{N}$,

$$C_p(B, C_q(A^{\natural}_B)) = C_q(A^{\natural}_B) \otimes B^{\otimes p} = B \otimes A^{\otimes(q+1)} \otimes B^{\otimes p},$$

the differential $\delta : C_p(B, C_q(A^{\natural}_B)) \rightarrow C_{p-1}(B, C_q(A^{\natural}_B))$ is

$$\begin{aligned} \delta(b_0 | a_0, \dots, a_q | b_1, \dots, b_p) &= ((b_0 | a_0, \dots, a_q).b_1 | b_2, \dots, b_p) \\ &+ \sum_{i=1}^{p-1} (-1)^i (b_0 | a_0, \dots, a_q | b_1, \dots, b_i b_{i+1}, \dots, b_p) \\ &+ (-1)^p (b_p.(b_0 | a_0, \dots, a_q) | b_1, \dots, b_{p-1}). \end{aligned} \tag{11}$$

Denote this Hochschild homology by $H_\bullet(B, C_q(A^{\natural}_B))$.

A difference occurs here, as the positions of A 's and B 's are changed.

Corollary 4.9. *$C_\bullet(B, C_\bullet(A^{\natural}_B))$ is a cylindrical module, which is isomorphic to $A\natural B$.*

Proof. We give the isomorphisms between $C_\bullet(B, C_\bullet(A^{\natural}_B))$ and $A\natural B$, then the bi-paracyclic operators on $C_\bullet(B, C_\bullet(A^{\natural}_B))$ are constructed from the operators on $A\natural B$ through the isomorphisms. In this way, we get the corollary. We give the isomorphisms and the operators on $C_\bullet(B, C_\bullet(A^{\natural}_B))$ explicitly. For each $p, q \in \mathbb{N}$,

$$\phi_{p,q} : C_p(B, C_q(A^{\natural}_B)) \longrightarrow A\natural B(p, q)$$

$$\phi_{p,q} = (id^{\otimes p} \otimes \Theta_q^{-1})(id^{\otimes(p-1)} \otimes \Theta_q^{-1} \otimes id) \cdots (id \otimes \Theta_q^{-1} \otimes id^{\otimes(p-1)}),$$

and

$$\psi_{p,q} : A\natural B(p, q) \longrightarrow C_p(B, C_q(A^{\natural}_B))$$

$$\psi_{p,q} = \phi_{p,q}^{-1} = (id \otimes \Theta_q \otimes id^{\otimes(p-1)})(id^{\otimes 2} \otimes \Theta_q \otimes id^{\otimes(p-2)}) \dots (id^{\otimes p} \otimes \Theta_q).$$

We can see that

$$\phi\delta = b\phi, \quad \psi b = \delta\psi.$$

Hence, the operators $(\mathbf{d}_i, \mathbf{s}_j, \mathbf{t}, \bar{\mathbf{d}}_i, \bar{\mathbf{s}}_j, \bar{\mathbf{t}})$ on $C_p(B, C_q(A_B^{\natural}))$ are defined as follows:

$$(12) \quad \begin{aligned} \mathbf{d}_i^{p,q} &= \psi_{p-1,q} d_i^{p,q} \phi_{p,q}, & \bar{\mathbf{d}}_i^{p,q} &= \psi_{p,q-1} \bar{d}_i^{p,q} \phi_{p,q}, \\ \mathbf{s}_j^{p,q} &= \psi_{p+1,q} s_j^{p,q} \phi_{p,q}, & \bar{\mathbf{s}}_j^{p,q} &= \psi_{p,q+1} \bar{s}_j^{p,q} \phi_{p,q}, \\ \mathbf{t}_{p,q} &= \psi_{p,q} t_{p,q} \phi_{p,q}, & \bar{\mathbf{t}}_{p,q} &= \psi_{p,q} \bar{t}_{p,q} \phi_{p,q}. \end{aligned}$$

□

Define $C_{\bullet}^B(A_B^{\natural})$ as the *co-invariant space* of $C_{\bullet}(A_B^{\natural})$ under the left and right actions given in Lemma 4.8, i.e.,

$$C_{\bullet}^B(A_B^{\natural}) = C_{\bullet}(A_B^{\natural}) / \text{span}\{b.x - x.b \mid b \in B, x \in C_{\bullet}(A_B^{\natural})\}.$$

And we define the following operators on $C_{\bullet}^B(A_B^{\natural})$:

$$(13) \quad \begin{aligned} \tau'_n(b \mid a_0, \dots, a_n) &= (R^{-1}(a_n \otimes b), a_0, \dots, a_{n-1}) \\ \partial'_i(b \mid a_0, \dots, a_n) &= (b \mid a_0, \dots, a_i a_{i+1}, \dots, a_n), \quad \text{for } 0 \leq i < n, \\ \partial'_n(b \mid a_0, \dots, a_n) &= \partial'_0 \tau'_n(b \mid a_0, \dots, a_n), \\ \sigma'_j(b \mid a_0, \dots, a_n) &= (b \mid a_0, \dots, a_j, 1, \dots, a_n), \quad \text{for } 0 \leq j \leq n. \end{aligned}$$

Indeed, τ'_n is induced by $\bar{t}_{0,n}$, as $\psi_{p,q} = id$ and $\phi_{p,q} = id$ when p is 0. One can check that these operators are well defined on the co-invariant space.

Theorem 4.10. *For any $p \in \mathbb{N}$, $H_p(B, C_{\bullet}(A_B^{\natural}))$ is a cyclic module with $(\bar{\mathbf{d}}_i, \bar{\mathbf{s}}_j, \bar{\mathbf{t}})$ induced from operators of $C_{\bullet}(B, C_{\bullet}(A_B^{\natural}))$ defined in (12). Especially, we have*

$$H_0(B, C_{\bullet}(A_B^{\natural})) = C_{\bullet}^B(A_B^{\natural})$$

is a cyclic module with operators defined in (13).

We define a filtration on $\text{Tot}(A_{\natural}B) \boxtimes W$ by columns. Set

$$\tilde{F}_n^q(\text{Tot}(A_{\natural}B) \boxtimes W) = \sum_{\substack{i+j=n+2l, \\ j \leq q+2l, \\ l \geq 0}} (B^{\otimes(i+1)} \otimes A^{\otimes(j+1)}) u^l \otimes_{k[u]} W, \quad \text{for } q \geq 0;$$

and $\tilde{F}_n^q(\text{Tot}(A_{\natural}B) \boxtimes W) = 0$ for $q < 0$.

The spectral sequence $\tilde{E}_{q,p}^r$ of this filtration with $\tilde{d}^r : \tilde{E}_{q,p}^r \rightarrow \tilde{E}_{q-r,p+r-1}^r$ starts from

$$\tilde{E}_{q,p}^0 = \sum_{l \geq 0} (B^{\otimes(p+1)} \otimes A^{\otimes(q+2l+1)}) u^l \otimes_{k[u]} W,$$

equipped with $\tilde{d}^0 = b \otimes id : \tilde{E}_{q,p}^0 \longrightarrow \tilde{E}_{q,p-1}^0$.

Lemma 4.11. *The E^1 -term of the spectral sequence is*

$$\tilde{E}_{\bullet,p}^1 = H_p(B, C_\bullet(A_B^\natural)) \boxtimes W,$$

equipped with $\tilde{d}^1 : \tilde{E}_{q,p}^1 \longrightarrow \tilde{E}_{q-1,p}^1$ that is induced by $(\bar{b} + u\bar{B}) \otimes id$.

Theorem 4.12. *The E^2 -term of the spectral sequence is identified with the cyclic homology of the cyclic module $H_\bullet(B, C_\bullet(A_B^\natural))$ with coefficients in W . It converges to the cyclic homology of the strong smash product algebra $A \#_R B$ with coefficients in W . That is,*

$$\tilde{E}_{q,p}^2 \cong HC_q(H_p(B, C_\bullet(A_B^\natural)); W) \Rightarrow HC_{p+q}(A \#_R B; W).$$

By Proposition 1.1.13 of [17], we can use the derived functor Tor to express the Hochschild homology of an algebra A with coefficients in M which is an A -bimodule, that is,

$$H_n(A, M) \cong \text{Tor}_n^{A^e}(M, A),$$

where $A^e = A \otimes A^{op}$. For a separable algebra that is projective over its enveloping algebra, its homology with coefficients in any module is zero. Hence, the spectral sequence collapses at E^2 , that is, $E_{p,q}^2 = 0$ for all p, q unless $q = 0$. So we have

Corollary 4.13. *If the algebra A (resp. B) is separable, then there is a natural isomorphism of cyclic homology groups*

$$\begin{aligned} HC_n(A \#_R B; W) &\cong HC_n(C_\bullet^A(A_B^\natural); W), \\ (\text{resp.}, \quad HC_n(A \#_R B; W) &\cong HC_n(C_\bullet^B(A_B^\natural); W)). \end{aligned}$$

From the above results, one can observe that our theorems take advantage of good homological property of either of two subalgebras. Even in the case of the crossed product algebra $A \rtimes H$, where H is a Hopf algebra with invertible antipode and A is an H -module algebra, the “nice” homological property of A sometimes will play a key role in computing the cyclic homology of the crossed product, by comparison with the homological property of H being weak. We will illustrate this point by examples in the next section.

5 Examples

5.1 In this subsection, we apply our theorems to Majid’s double crossproduct of Hopf algebras which is inspired by bismash product of groups defined by Takeuchi [26]. Bismash product of groups is a generalization of semiproduct of groups. In order to define this product, he provided the notion of a matched pair of groups. Given a matched pair of groups (G, K) , the bismash product of G and K denoted by $G \bowtie K$ is still a group.

The theory is developed by Majid [18]. He defined a matched pair of Hopf algebras and constructed a product Hopf algebra which he called a double crossproduct of Hopf algebras. Using this new definition he provided another way to construct Drinfeld’s quantum double. We start by recalling the definition due to Majid [18].

Definition 5.1 ([18]). A pair (B, H) of Hopf algebras is said to be *matched* if B is a left H -module coalgebra via α , and H is a right B -module coalgebra via β ,

$$\alpha : H \otimes B \rightarrow B, \quad \alpha(h \otimes b) = h \triangleright b, \quad \beta : H \otimes B \rightarrow H, \quad \beta(h \otimes b) = h \triangleleft b,$$

such that the following equalities hold for $\forall b, c \in B, h, g \in H$.

$$(14) \quad h \triangleright 1_B = \varepsilon_H(h)1_B, \quad h \triangleright (bc) = \sum \left(h_{(1)} \triangleright b_{(1)} \right) \left((h_{(2)} \triangleleft b_{(2)}) \triangleright c \right)$$

$$(15) \quad 1_H \triangleleft b = 1_H \varepsilon_B(b), \quad (hg) \triangleleft b = \sum \left(h \triangleleft (g_{(1)} \triangleright b_{(1)}) \right) \left(g_{(2)} \triangleleft b_{(2)} \right)$$

$$(16) \quad \sum h_{(1)} \triangleleft b_{(1)} \otimes h_{(2)} \triangleright b_{(2)} = h_{(2)} \triangleleft b_{(2)} \otimes h_{(1)} \triangleright b_{(1)}.$$

The *double crossproduct* $B \bowtie H$ is a Hopf algebra equipped with

$$\begin{aligned} (b \otimes h)(c \otimes g) &= b(h_{(1)} \triangleright c_{(1)}) \otimes (h_{(2)} \triangleleft c_{(2)})g \\ \Delta(b \otimes h) &= b_{(1)} \otimes h_{(1)} \otimes b_{(2)} \otimes h_{(2)} \\ \varepsilon(b \otimes h) &= \varepsilon_B(b)\varepsilon_H(h) \\ S(b \otimes h) &= (1 \otimes S_H h)(S_B b \otimes 1). \end{aligned}$$

The double crossproduct of Hopf algebras relates closely to the smash product algebra.

Proposition 5.2. *Let (B, H) be a matched pair of Hopf algebras. If H and B have invertible antipodes, then the double crossproduct of B and H denoted by $B \bowtie H$ is a strong smash product algebra. In particular, the group algebra of the bismash product of a matched pair of groups is a strong smash product algebra.*

We need the following lemma in the proof of Proposition 5.2.

Lemma 5.3. *Let (B, H) be a matched pair of Hopf algebras. If both H and B have invertible antipodes, then we have the following identities:*

$$(17) \quad S_B^{-1}(h \triangleright b) = (h \triangleleft b_{(2)}) \triangleright S_B^{-1}(b_{(1)}),$$

$$(18) \quad S_H^{-1}(h \triangleleft b) = S_H^{-1}(h_{(2)}) \triangleleft (h_{(1)} \triangleright b).$$

Proof. Since $(B, \eta, m, \Delta, \varepsilon_B, S_B)$ is a Hopf algebra, then $(B^{op}, \eta, m^{op}, \Delta, \varepsilon_B, S_B^{-1})$ is also a Hopf algebra. Denote the convolution map on $\text{Hom}(B^{op}, B^{op})$ by $*'$. Define the operator $T \in \text{End}_k(H \triangleright B)$ by

$$T(h \triangleright b) := (h \triangleleft b_{(2)}) \triangleright S_B^{-1}(b_{(1)}).$$

We should check that

$$(id *' T)(h \triangleright b) = \varepsilon_B(h \triangleright b)1_B \quad \text{and} \quad (T *' id)(h \triangleright b) = \varepsilon_B(h \triangleright b)1_B.$$

Actually, we only need to check the first equality. Indeed, if it holds, then

$$\begin{aligned} T(h \triangleright b) &= (\varepsilon_B *' T)(h \triangleright b) = ((S_B^{-1} *' id) *' T)(h \triangleright b) \\ &= (S_B^{-1} *' (id *' T))(h \triangleright b) = (S_B^{-1} *' \varepsilon_B)(h \triangleright b) = S_B^{-1}(h \triangleright b). \end{aligned}$$

From (14) and (16), we have

$$\begin{aligned} (id *' T)(h \triangleright b) &= (h_{(2)} \triangleright b_{(2)})T(h_{(1)} \triangleright b_{(1)}) \\ &= (h_{(2)} \triangleright b_{(3)})((h_{(1)} \triangleleft b_{(2)}) \triangleright S_B^{-1}(b_{(1)})) \\ &= (h_{(1)} \triangleright b_{(2)})((h_{(2)} \triangleleft b_{(3)}) \triangleright S_B^{-1}(b_{(1)})) \\ &= h \triangleright (b_{(2)} S_B^{-1}(b_{(1)})) = \varepsilon_B(b)h \triangleright 1_B = \varepsilon_H(h)\varepsilon_B(b)1_B \\ &= \varepsilon_B(h \triangleright b)1_B. \end{aligned}$$

Using the same method, we can prove (18). □

Proof of Proposition 5.2. We need to construct an isomorphism R from $H \otimes B$ to $B \otimes H$, which is quasitriangular and normal. For $b \in B, h \in H$, set

$$R(h \otimes b) = \sum h_{(1)} \triangleright b_{(1)} \otimes h_{(2)} \triangleleft b_{(2)}.$$

1) R is quasitriangular: $\forall h, g \in H, b \in B$,

$$\begin{aligned} (id \otimes m)R_{12}R_{23}(h \otimes g \otimes b) &= \sum (id \otimes m)R_{12}(h \otimes g_{(1)} \triangleright b_{(1)} \otimes g_{(2)} \triangleleft b_{(2)}) \\ &= h_{(1)} \triangleright (g_{(1)} \triangleright b_{(1)})_{(1)} \otimes (h_{(2)} \triangleleft (g_{(1)} \triangleright b_{(1)})_{(2)}) (g_{(2)} \triangleleft b_{(2)}) \\ &= h_{(1)} \triangleright (g_{(1)} \triangleright b_{(1)}) \otimes (h_{(2)} \triangleleft (g_{(2)} \triangleright b_{(2)})) (g_{(3)} \triangleleft b_{(3)}) \\ &= \sum (h_{(1)}g_{(1)}) \triangleright b_{(1)} \otimes (h_{(2)}g_{(2)}) \triangleleft b_{(2)} = R(hg \otimes b). \end{aligned}$$

The third equality holds due to the H -module coalgebra structure of B , the forth equality holds because of (15). Similarly, one can prove that $R \circ (id \otimes m) = (m \otimes id)R_{23}R_{12}$.

2) R is normal: $\forall h \in H$,

$$R(h \otimes 1_B) = \sum h_{(1)} \triangleright 1_B \otimes h_{(2)} \triangleleft 1_B = \varepsilon(h_{(1)})1_B \otimes h_{(2)} = 1_B \otimes h.$$

Similarly, one can prove that $R(1_H \otimes b) = b \otimes 1_H$.

3) R is invertible: For $\forall b \in B, h \in H$, set $r : B \otimes H \rightarrow H \otimes B$

$$r(b \otimes h) := h_{(3)} \triangleleft (S_H^{-1}(h_{(2)}) \triangleright S_B^{-1}(b_{(3)})) \otimes (S_H^{-1}(h_{(1)}) \triangleleft S_B^{-1}(b_{(2)})) \triangleright b_{(1)}.$$

We claim that r is the inverse of R .

$$\begin{aligned} R \circ r(b \otimes h) &= R(h_{(3)} \triangleleft (S_H^{-1}(h_{(2)}) \triangleright S_B^{-1}(b_{(3)})) \otimes (S_H^{-1}(h_{(1)}) \triangleleft S_B^{-1}(b_{(2)})) \triangleright b_{(1)}) \\ &= (h_{(3)} \triangleleft (S_H^{-1}(h_{(2)}) \triangleright S_B^{-1}(b_{(3)})))_{(1)} \triangleright ((S_H^{-1}(h_{(1)}) \triangleleft S_B^{-1}(b_{(2)})) \triangleright b_{(1)})_{(1)} \\ &\quad \otimes (h_{(3)} \triangleleft (S_H^{-1}(h_{(2)}) \triangleright S_B^{-1}(b_{(3)})))_{(2)} \triangleleft ((S_H^{-1}(h_{(1)}) \triangleleft S_B^{-1}(b_{(2)})) \triangleright b_{(1)})_{(2)} \\ &= (h_{(5)} \triangleleft (S_H^{-1}(h_{(4)}) \triangleright S_B^{-1}(b_{(6)}))) \triangleright ((S_H^{-1}(h_{(2)}) \triangleleft S_B^{-1}(b_{(4)})) \triangleright b_{(1)}) \\ &\quad \otimes (h_{(6)} \triangleleft (S_H^{-1}(h_{(3)}) \triangleright S_B^{-1}(b_{(5)}))) \triangleleft ((S_H^{-1}(h_{(1)}) \triangleleft S_B^{-1}(b_{(3)})) \triangleright b_{(2)}) \\ &= ((h_{(5)} \triangleleft (S_H^{-1}(h_{(4)}) \triangleright S_B^{-1}(b_{(6)}))) (S_H^{-1}(h_{(2)}) \triangleleft S_B^{-1}(b_{(4)}))) \triangleright b_{(1)} \\ &\quad \otimes h_{(6)} \triangleleft ((S_H^{-1}(h_{(3)}) \triangleright S_B^{-1}(b_{(5)}))) ((S_H^{-1}(h_{(1)}) \triangleleft S_B^{-1}(b_{(3)})) \triangleright b_{(2)})) \\ &= ((h_{(5)} \triangleleft (S_H^{-1}(h_{(4)}) \triangleright S_B^{-1}(b_{(6)}))) (S_H^{-1}(h_{(3)}) \triangleleft S_B^{-1}(b_{(5)}))) \triangleright b_{(1)} \\ &\quad \otimes h_{(6)} \triangleleft ((S_H^{-1}(h_{(2)}) \triangleright S_B^{-1}(b_{(4)}))) ((S_H^{-1}(h_{(1)}) \triangleleft S_B^{-1}(b_{(3)})) \triangleright b_{(2)})) \\ &= ((h_{(3)} S_H^{-1}(h_{(2)})) \triangleleft S_B^{-1}(b_{(4)})) \triangleright b_{(1)} \otimes h_{(4)} \triangleleft (S_H^{-1}(h_{(1)}) \triangleright (S_B^{-1}(b_{(3)}) b_{(2)})) \\ &= (1_H \triangleleft S_B^{-1}(b_{(2)})) \triangleright b_{(1)} \otimes h_{(2)} \triangleleft (S_H^{-1}(h_{(1)}) \triangleright 1_B) \\ &= b \otimes h. \end{aligned}$$

The third and the forth equalities above are due to the B -module coalgebra structure of H and the H -module coalgebra structure of B . The fifth equality is due to (16). The sixth and the last equalities hold because of (14) and (15).

$$r \circ R(h \otimes b) = r(\sum h_{(1)} \triangleright b_{(1)} \otimes h_{(2)} \triangleleft b_{(2)})$$

$$\begin{aligned}
&= (h_{(2)} \triangleleft b_{(2)})_{(3)} \triangleleft \left(S_H^{-1}((h_{(2)} \triangleleft b_{(2)})_{(2)}) \triangleright S_B^{-1}((h_{(1)} \triangleright b_{(1)})_{(3)}) \right) \\
&\quad \otimes \left(S_H^{-1}((h_{(2)} \triangleleft b_{(2)})_{(1)}) \triangleleft S_B^{-1}((h_{(1)} \triangleright b_{(1)})_{(2)}) \right) \triangleright (h_{(1)} \triangleright b_{(1)})_{(1)} \\
&= h_{(6)} \triangleleft b_{(6)} \left(S_H^{-1}(h_{(5)} \triangleleft b_{(5)}) \triangleright S_B^{-1}(h_{(3)} \triangleright b_{(3)}) \right) \\
&\quad \otimes \left(S_H^{-1}(h_{(4)} \triangleleft b_{(4)}) \triangleleft S_B^{-1}(h_{(2)} \triangleright b_{(2)}) \right) h_{(1)} \triangleright b_{(1)} \\
&= h_{(6)} \triangleleft b_{(6)} \left(S_H^{-1}(h_{(5)} \triangleleft b_{(5)}) \triangleright S_B^{-1}(h_{(4)} \triangleright b_{(4)}) \right) \\
&\quad \otimes \left(S_H^{-1}(h_{(3)} \triangleleft b_{(3)}) \triangleleft S_B^{-1}(h_{(2)} \triangleright b_{(2)}) \right) h_{(1)} \triangleright b_{(1)} \\
&= h_{(7)} \triangleleft b_{(7)} \left(S_H^{-1}(h_{(6)} \triangleleft b_{(6)})(h_{(5)} \triangleleft b_{(5)}) \triangleright S_B^{-1}(b_{(4)}) \right) \\
&\quad \otimes \left(S_H^{-1}(h_{(4)} \triangleleft b_{(4)}) \triangleleft (h_{(3)} \triangleright b_{(3)}) S_B^{-1}(h_{(2)} \triangleright b_{(2)}) \right) h_{(1)} \triangleright b_{(1)} \\
&= h_{(3)} \triangleleft b_{(3)} S_B^{-1}(b_{(2)}) \otimes S_H^{-1}(h_{(2)}) h_{(1)} \triangleright b_{(1)} \\
&= h \otimes b.
\end{aligned}$$

The fifth equality above holds because of (17) and (18).

For (G, K) a matched pair of groups, since $k[G \bowtie K] = k[G] \bowtie k[K]$, it is a strong smash product algebra. \square

Therefore, thanks to Theorems 4.7 and 4.12, we have

Corollary 5.4. *Given a matched pair of Hopf algebra (B, H) , if each of B and H has an invertible antipode, then*

$$\begin{aligned}
\mathrm{HC}_p\left(\mathrm{H}_q(B, C_\bullet({}_B^{\natural}H)); W\right) &\Rightarrow \mathrm{HC}_{p+q}(B \bowtie H; W), \\
\mathrm{HC}_q\left(\mathrm{H}_p(H, C_\bullet(B_H^{\natural})); W\right) &\Rightarrow \mathrm{HC}_{p+q}(B \bowtie H; W).
\end{aligned}$$

For a finite group G and an arbitrary G -bimodule M , since $k[G]$ is semisimple, $\mathrm{H}_n(k[G], M)$ is 0 for all n except for $n = 0$. Then by the above corollary, Theorems 4.4 and 4.10, we have

Corollary 5.5. *Given a matched pair of finite groups (G, K) , then*

$$\begin{aligned}
\mathrm{HC}_n(k[G \bowtie K]; W) &\cong \mathrm{HC}_n(C_\bullet^G({}_G^{\natural}K); W), \\
\mathrm{HC}_n(k[G \bowtie K]; W) &\cong \mathrm{HC}_n(C_\bullet^K(G_K^{\natural}); W).
\end{aligned}$$

If H is a finite dimensional Hopf algebra, then the antipode of H is always invertible (see, Corollary 5.6.1 of [23]). Using the adjoint action of H on itself, Majid in Example 4.6 of [18] constructed a matched pair (H, H^{*cop}) and deduced the Drinfeld's quantum double $D(H) = H^{*cop} \bowtie H$. By Corollaries 5.4 and 5.5, we have

Corollary 5.6. *If H is a finite dimensional Hopf algebra, then*

$$\mathrm{HC}_q\left(\mathrm{H}_p(H, C_\bullet(H^{*cop}_H{}^{\natural})); W\right) \Rightarrow \mathrm{HC}_{p+q}(D(H); W).$$

If moreover, H is semisimple (equivalently, there is an integral $t \in H$ with $\varepsilon(t) = 1$), then

$$\mathrm{HC}_n(D(H); W) \cong \mathrm{HC}_n(C_\bullet^H(H^{*cop}_H{}^{\natural}); W).$$

Remark 5.7. Actually, any Drinfeld's quantum double turns out to be of Majid's double crossproduct structure (see [19]), while the recently appeared attractive objects, such as the two-parameter or the multiparameter (restricted) quantum (affine) groups, the pointed Hopf algebras arising from Nichols algebras of diagonal type (cf. [4, 5, 12, 13, 14, 22, 2, 3, 10] and references therein), are of Drinfeld's double structures (under certain conditions for the root of unity cases). Thereby, our machinery established for the strong smash product algebras is indeed suitable to a large class of many interesting Hopf algebras.

5.2 The following first example comes from the rank 1 case (modified) of the smash product algebra $\mathcal{A}_q \# \mathcal{D}_q$ introduced in [11] (p.525, subsection 3.5), which was used to define intrinsically and construct a quantum Weyl algebra $\mathcal{W}_q(2n)$. Although our example here is still a crossed product algebra, Proposition 5.3 in [1], under the assumption of the Hopf algebra H being semisimple (so automatically finite dimensional), does not work for our example.

Example 5.8. Let $q \in k$ be an N -th primitive root of unity. Define \mathcal{A} to be $k[x]/(x^N - 1)$ which is isomorphic to the group algebra $k[\mathbb{Z}/N\mathbb{Z}]$. Define \mathcal{D} to be the associative k -algebra generated by $\partial, \sigma^{\pm 1}$, subject to relation $\sigma^{-1}\partial\sigma = q\partial$. \mathcal{D} is a Hopf algebra with the coproduct, counit, and antipode defined as follows:

$$\begin{aligned}\Delta(\partial) &= \partial \otimes 1 + \sigma \otimes \partial, & \Delta(\sigma) &= \sigma \otimes \sigma, & \varepsilon(\partial) &= 0, & \varepsilon(\sigma) &= 1, \\ S(\partial) &= -\sigma^{-1}\partial, & S(\sigma) &= \sigma^{-1}.\end{aligned}$$

The antipode of \mathcal{D} is invertible, as $S^{2N} = id$. One can calculate that $S^{-1}(\partial) = -\partial\sigma^{-1}$ and $S^{-1}(\sigma) = \sigma^{-1}$. Let $(n)_q = 1 + q + \cdots + q^{n-1}$ for $0 < n \in \mathbb{N}$. Then $(N)_q = 0$.

Lemma 5.9. ([11]) \mathcal{A} is a \mathcal{D} -module algebra via $\partial.1 = 0, \sigma.1 = 1$, and for $n > 0$,

$$\partial.x^n = (n)_q x^{n-1}, \quad \sigma.x^n = q^n x^n.$$

Proof. We should first check that \mathcal{A} is a \mathcal{D} -module. Indeed, if $n > 0$,

$$\begin{aligned}(\sigma^{-1}\partial\sigma).x^n &= q(n)_q x^{n-1} = q\partial.x^n, \\ \partial.x^N &= (N)_q x^{N-1} = 0 = \partial.1, \text{ and } \sigma.x^N = q^N x^N = 1.\end{aligned}$$

From direct calculation, we get

$$(\partial.x^i)x^j + (\sigma.x^i)(\partial.x^j) = (i+j)_q x^{i+j-1} = \partial.(x^i x^j).$$

This completes the proof. □

As we stated in Example 1.5, the crossed product $\mathcal{A} \rtimes \mathcal{D}$ is a strong smash product algebra, since \mathcal{D} is a Hopf algebra with invertible antipode. For example, $R(\partial \otimes x^n) = (n)_q x^{n-1} \otimes 1 + q^n x^n \otimes \partial$, $R(\sigma \otimes x^n) = q^n x^n \otimes \sigma$ and $R^{-1}(x^n \otimes \partial) = q^{-n} \partial \otimes x^n - q^{-n}(n)_q \otimes x^{n-1}$ for $n > 0$.

Since \mathcal{A} is a group algebra of a finite group, then it is a semisimple Hopf algebra, so the spectral sequence collapses and we have

Corollary 5.10.

$$\mathrm{HC}_n(\mathcal{A} \rtimes \mathcal{D}; W) \cong \mathrm{HC}_n(C_{\bullet}^{\mathcal{A}}(\mathcal{D}); W).$$

Example 5.11. Let \mathcal{D}_N be the quotient algebra of \mathcal{D} by the ideal $\langle \partial^N \rangle$. As $\langle \partial^N \rangle$ is a Hopf ideal (owing to $\Delta(\partial^s) = \sum_{i=0}^s \binom{s}{i}_q \sigma^i \partial^{s-i} \otimes \partial^i$ and $\Delta(\partial^N) = \partial^N \otimes 1 + \sigma^N \otimes \partial^N$), \mathcal{D}_N is a Hopf algebra. In particular, when $N = 2$, \mathcal{D}_2 is nothing but the Pareigis' Hopf algebra \mathcal{P} (see [21] or the next subsection for definition).

Furthermore, consider the quotient Hopf algebra $\bar{\mathcal{D}}_N$ of \mathcal{D}_N by the Hopf ideal $\langle \sigma^N - 1 \rangle$. $\bar{\mathcal{D}}_N$ is just the Taft algebra. Cyclic homology of the Taft algebra as a special truncated quiver algebra is computed by Taillefer [24].

5.3 This subsection is devoted to effectively computing the cyclic homology of the Pareigis' Hopf algebra \mathcal{P} using our theory.

In [21], Pareigis defined a noncommutative and noncocommutative Hopf algebra \mathcal{P} , which links closely the category of complexes and the category of comodules over \mathcal{P} . That is, the category of complexes is equivalent as a tensor category to the category of comodules over \mathcal{P} . Explicitly, \mathcal{P} is defined to be the quotient algebra of the free algebra $k\langle s, t, t^{-1} \rangle$ by the two sided ideal that is generated by

$$tt^{-1} - 1, \quad t^{-1}t - 1, \quad s^2, \quad st + ts.$$

Then \mathcal{P} turns out to be a Hopf algebra with the following coproduct, counit and antipode,

$$\begin{aligned} \Delta(t) &= t \otimes t, \quad \varepsilon(t) = 1, \quad S(t) = t^{-1}; \\ \Delta(s) &= s \otimes 1 + t^{-1} \otimes s, \quad \varepsilon(s) = 0, \quad S(s) = st. \end{aligned}$$

\mathcal{P} can be regarded as the crossed product algebra of $k[s]/s^2$ and $k[t, t^{-1}]$, where $k[s]/s^2$ is a module algebra over $k[t, t^{-1}]$ with the conjugate action $t.s = tst^{-1} = -s$. Denote by D the algebra of dual number $k[s]/s^2$, and by T the Laurent polynomial ring $k[t, t^{-1}]$.

$$\mathcal{P} \cong D \rtimes T.$$

\mathcal{P} is a strong smash product algebra $D \#_R T$ with the invertible $R : T \otimes D \rightarrow D \otimes T$ defined to be

$$R(t^r \otimes s) = (-1)^r (s \otimes t^r).$$

Let $W = k[u]/uk[u]$. We would like to calculate the Hochschild homology of \mathcal{P} first. Consider the cyclic module $E_{\bullet, q}^1 = H_q(D, C_{\bullet}(D \natural T)) \cong \text{Tor}_q^{D^e}(D, C_{\bullet}(D \natural T))$. Its face maps, degeneracy maps, and cyclic operators are induced by the corresponding operators defined in (7) for the cylindrical module $D \natural T(\bullet, q)$.

Using the following resolution of D by projective D^e -modules (see e.g., [28])

$$R_{\bullet} : \cdots \xrightarrow{\nu} D^e \xrightarrow{\mu} D^e \xrightarrow{\nu} D^e \xrightarrow{\mu} D^e \xrightarrow{m} D \longrightarrow 0,$$

where $\mu = 1 \otimes s - s \otimes 1$, $\nu = 1 \otimes s + s \otimes 1$, m is the product of D , we get

$$H_q(D, C_p(D \natural T)) \cong \begin{cases} T_{ev}^{\otimes(p+1)} \otimes 1 \oplus T_{ev}^{\otimes(p+1)} \otimes s & \text{for } q = 0 \\ T_{ev}^{\otimes(p+1)} \otimes 1 \oplus T_{od}^{\otimes(p+1)} \otimes s & \text{for } q = 2n - 1 > 0, \\ T_{od}^{\otimes(p+1)} \otimes 1 \oplus T_{ev}^{\otimes(p+1)} \otimes s & \text{for } q = 2n > 0 \end{cases}$$

where

$$\begin{aligned} T_{ev}^{\otimes(p+1)} &:= k\{(t^{r_0}, \dots, t^{r_p}) \mid r_0 + \dots + r_p \text{ is even}\}, \\ T_{od}^{\otimes(p+1)} &:= k\{(t^{r_0}, \dots, t^{r_p}) \mid r_0 + \dots + r_p \text{ is odd}\}. \end{aligned}$$

In order to specify the operators of the cyclic module $H_q(D, C_\bullet(D^{\natural}T))$, we should represent the elements of $H_q(D, C_\bullet(D^{\natural}T))$ by elements of $D^{\natural}T(\bullet, q)$. According to the Comparison Theorem, there is a unique chain map lifting id_D from the resolution R_\bullet to the bar resolution of D up to chain homotopy equivalence. This required chain map ζ_\bullet is defined as follows,

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\nu} & D^e & \xrightarrow{\mu} & D^e & \xrightarrow{\nu} & D^e & \xrightarrow{\mu} & D^e & \xrightarrow{m} & D & \longrightarrow & 0 \\ & & \downarrow \zeta_3 & & \downarrow \zeta_2 & & \downarrow \zeta_1 & & \downarrow \zeta_0 & & \downarrow id & & \\ \cdots & \xrightarrow{b'} & D^{\otimes 5} & \xrightarrow{b'} & D^{\otimes 4} & \xrightarrow{b'} & D^{\otimes 3} & \xrightarrow{b'} & D^{\otimes 2} & \xrightarrow{b'} & D & \longrightarrow & 0 \end{array}$$

where $\zeta_n : D^e \rightarrow D^{\otimes(n+2)}$ and

$$\begin{aligned} \zeta_0 &= id, \\ \zeta_n(s \otimes s) &= s^{\otimes(n+2)}, \quad \zeta_n(1 \otimes 1) = -1 \otimes s^{\otimes n} \otimes 1, \\ \zeta_n(1 \otimes s) &= (-1)^n \otimes s^{\otimes(n+1)}, \quad \zeta_n(s \otimes 1) = (-1)^n s^{\otimes(n+1)} \otimes 1. \end{aligned}$$

Hence,

$$E_{p,q}^1 = H_q(D, C_p(D^{\natural}T)) \cong \begin{cases} T^{\otimes(p+1)} \otimes 1 \oplus T_{ev}^{\otimes(p+1)} \otimes s & \text{for } q = 0 \\ T_{ev}^{\otimes(p+1)} \otimes 1 \otimes s^{\otimes(2n-1)} \oplus T_{od}^{\otimes(p+1)} \otimes s^{\otimes 2n} & \text{for } q = 2n - 1 > 0. \\ T_{od}^{\otimes(p+1)} \otimes 1 \otimes s^{\otimes 2n} \oplus T_{ev}^{\otimes(p+1)} \otimes s^{\otimes(2n+1)} & \text{for } q = 2n > 0 \end{cases}$$

The cyclic operator τ on $H_q(D, C_p(D^{\natural}T))$ is defined via

$$\tau(t^{r_0}, \dots, t^{r_p} \mid s^l \mid \underbrace{s, \dots, s}_{n \text{ times}}) = (-1)^{(l+n)r_p} (t^{r_p}, t^{r_0}, \dots, t^{r_{p-1}} \mid s^l \mid \underbrace{s, \dots, s}_{n \text{ times}}), \text{ where } l = 0, 1.$$

We can describe the cyclic modules $E_{\bullet,q}^1$ simply. Let $C_\bullet(T)$ be the cyclic module of the algebra T . Since the face maps, degeneracy maps, and the cyclic operators of $C_\bullet(T)$ do not change the total degree of t , $C_\bullet(T)$ can be decomposed into the direct sum of two sub-cyclic modules $C_\bullet(T)_{ev}$ and $C_\bullet(T)_{od}$ with $C_p(T)_{ev} = T_{ev}^{\otimes(p+1)}$ and $C_p(T)_{od} = T_{od}^{\otimes(p+1)}$. Let $C'_\bullet(T)_{ev}$ be the cyclic module with $C'_n(T)_{ev} = T_{ev}^{\otimes(n+1)}$ and the operators

$$\begin{aligned} \tau(t^{r_0}, t^{r_1}, \dots, t^{r_n}) &= (-1)^{r_n} (t^{r_n}, t^{r_0}, \dots, t^{r_{n-1}}), \\ \partial_i(t^{r_0}, t^{r_1}, \dots, t^{r_n}) &= (t^{r_0}, \dots, t^{r_i+r_{i+1}}, \dots, t^{r_n}), \quad \text{for } 0 \leq i < n, \\ \partial_n(t^{r_0}, t^{r_1}, \dots, t^{r_n}) &= (-1)^{r_n} (t^{r_0+r_n}, t^{r_1}, \dots, t^{r_{n-1}}), \\ \sigma_j(t^{r_0}, t^{r_1}, \dots, t^{r_n}) &= (t^{r_0}, \dots, t^{r_j}, 1, t^{r_{j+1}}, \dots, t^{r_n}), \quad \text{for } 0 \leq j \leq n, \end{aligned}$$

where $r_0 + \dots + r_n$ is an even integer.

Corollary 5.12. $E_{\bullet,0}^1$ is identified with $C_\bullet(T) \oplus C'_\bullet(T)_{ev}$ as cyclic modules; for $n > 0$, $E_{\bullet,n}^1$ is identified with $C_\bullet(T)_{od} \oplus C'_\bullet(T)_{ev}$ as cyclic modules.

Lemma 5.13. The Hochschild homology of $(C'_\bullet(T)_{ev}, \partial)$ is 0.

Proof. Indeed, we can construct a chain contraction $\{h_n : C'_n(T)_{ev} \rightarrow C'_{n+1}(T)_{ev}\}$ of the identity chain map. That is,

$$h_n(t^{r_0}, t^{r_1}, \dots, t^{r_n}) = \frac{1}{2} \sum_{i=0}^n (-1)^i (t^{r_0-1}, t^{r_1}, \dots, t^{r_i}, t, t^{r_{i+1}}, \dots, t^{r_n}).$$

We can check directly that $\partial h_n + h_{n-1} \partial = id$. Hence $H_n(C'_\bullet(T)_{ev}, \partial) = 0$, for all $n \geq 0$. \square

Since $\mathrm{HH}_n(T) \cong \begin{cases} T & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2 \end{cases}$ and $\mathrm{HH}_n(T)_{od} \cong \begin{cases} T_{od} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2 \end{cases}$, we obtain that

$$E_{p,q}^2 = 0, \forall p \neq 0, 1;$$

$$E_{0,0}^2 = E_{1,0}^2 = T \quad \text{and} \quad E_{0,q}^2 = E_{1,q}^2 = T_{od}, \text{ for } q > 0.$$

So the spectral sequence collapses at E^2 , and $\mathrm{HH}_n(\mathcal{P}) = \bigoplus_{p+q=n} E_{p,q}^2$.

Corollary 5.14. *The Hochschild homology of \mathcal{P} is*

$$\mathrm{HH}_n(\mathcal{P}) \cong \begin{cases} T & \text{for } n = 0 \\ T \oplus T_{od} & \text{for } n = 1 \\ T_{od} \oplus T_{od} & \text{for } n > 1 \end{cases}.$$

The cyclic homology of T is well-known (see e.g., p.337 in [28])

$$\mathrm{HC}_n(T) \cong \begin{cases} T & \text{for } n = 0 \\ k & \text{for } n > 0 \end{cases}.$$

Thanks to the short exact sequences $0 \rightarrow \overline{\mathrm{HC}}_{n-1}(\mathcal{P}) \rightarrow \overline{\mathrm{HH}}_n(\mathcal{P}) \rightarrow \overline{\mathrm{HC}}_n(\mathcal{P}) \rightarrow 0$ (see e.g., [17] Theorem 4.1.13), where $\overline{\mathrm{HH}}_n(\mathcal{P}) := \mathrm{HH}_n(\mathcal{P})/\mathrm{HH}_n(T)$ and $\overline{\mathrm{HC}}_n(\mathcal{P}) := \mathrm{HC}_n(\mathcal{P})/\mathrm{HC}_n(T)$, we get the cyclic homology of \mathcal{P} .

Proposition 5.15.

$$\mathrm{HC}_n(\mathcal{P}) \cong \begin{cases} T & \text{for } n = 0 \\ T_{od} \oplus k & \text{for } n > 0 \end{cases}.$$

Remark 5.16. The Laurent polynomial ring T is isomorphic to the group algebra $k[\mathbb{Z}]$. If making use of the results of [9], one can construct another spectral sequence $\tilde{E}_{p,q}^r$ with $\tilde{E}_{p,q}^2 = 0$ for $\forall q \neq 0, 1$, converging to the cyclic homology of \mathcal{P} . In this way, it remains to determine $d^2 : \tilde{E}_{p+2,0}^2 \rightarrow \tilde{E}_{p,1}^2$ to achieve \tilde{E}^3 . Since this spectral sequence collapses at \tilde{E}^3 , one then does more.

Acknowledgements. The authors are supported in part by the NNSF (Grants: 10971065, 10728102), the PCSIRT and the RFDP from the MOE, the National and Shanghai Leading Academic Discipline Projects (Project Number: B407). The first author would like to express her gratitude to Professor Ezra Getzler for pointing out the flatness condition. She is indebted to her advisor Professor Marc Rosso for his kind help. The authors also would like to thank Professor Joachim Cuntz for his useful comments and encouragement.

References

- [1] R. Akbarpour and M. Khalkhali, *Hopf algebra equivariant cyclic homology and cyclic homology of crossed product algebras*, J. reine angew. Math. **559** (2003), 137–152.
- [2] N. Andruskiewitsch, D. Raford and H. Schneider, *Complete reducibility theorems for modules over pointed Hopf algebras*, arXiv:1001.3177v1.
- [3] N. Andruskiewitsch and H. Schneider, *Pointed Hopf algebras*, New directions in Hopf algebras, 1-68, Math. Sci. Res. Inst. Publ. **43**, Cambridge Univ. Press, Cambridge, 2002.

- [4] G. Benkart and S. Witherspoon, *Two-parameter quantum groups (of type A) and Drinfel'd doubles*, *Algebr. Represent. Theory* **7** (2004), 261–286.
- [5] N. Bergeron, Y. Gao and N. Hu, *Drinfel'd doubles and Lusztig's symmetries of two-parameter quantum groups*, *J. Algebra* **301** (2006), 378–405.
- [6] J. L. Brylinski, *Cyclic homology and equivariant theories*, *Ann. Inst. Fourier.* **37** (1987), 15–28.
- [7] S. Caenepeel, I. Bogdan, G. Militaru and S. Zhu, *The factorization problem and the smash biproduct of algebras and coalgebras*, *Algebr. Represent. Theory*, **3** (2000), 19–42.
- [8] B. L. Feigin and B. L. Tsygan, *Additive K-theory, K-Theory, Arithmetic and Geometry*, *Lect. Notes Math.* **1289** (1986), 67–209.
- [9] E. Getzler and J. D. S. Jones, *The cyclic homology of crossed product algebras*, *J. reine angew. Math.* **445** (1993), 163–174.
- [10] I. Heckenberger, *Lusztig isomorphisms for Drinfel'd doubles of bosonizations of Nichols algebras of diagonal type*, *J. Algebra* **323** (8) (2010), 2130–2182.
- [11] N. Hu, *Quantum divided power algebra, q -derivatives, and some new quantum groups*, *J. Algebra* **232** (2000), 507–540.
- [12] N. Hu and Y. Pei, *Notes on two-parameter quantum groups, (I)*, *Sci. in China, Ser. A.* **51** (6) (2008), 1101–1110. (arXiv.math.QA/0702298).
- [13] N. Hu, M. Rosso and H. Zhang, *Two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{sl}}_n)$, Drinfel'd realization and quantum affine Lyndon basis*, *Comm. Math. Phys.* **278** (2008), 453–486.
- [14] N. Hu and X. Wang, *Convex PBW-type Lyndon bases and restricted two-parameter quantum groups of type B*, *J. Geom. Phys.* **60** (2010), 430–453.
- [15] C. Kassel, *Cyclic homology, comodules, and mixed complexes*, *J. Algebra* **107** (1987), 195–216.
- [16] M. Khalkhali, B. Rangipour, *On the generalized cyclic Eilenberg-Zilber theorem*, *Canad. Math. Bull.* **47** (2004), 38–48.
- [17] J. L. Loday, *Cyclic Homology*, *Grundlehren der Mathematischen Wissenschaften*, **301**, Springer-Verlag, Berlin, 1998.
- [18] S. Majid, *Physics for algebraists: noncommutative and noncocommutative Hopf algebras by a bicrossproduct construction*, *J. Algebra* **130** (1990), 17–64.
- [19] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, 1995.
- [20] V. Nistor, *Group homology and the cyclic homology of crossed products*, *Invent. Math.* **99** (1989), 411–424.
- [21] B. Pareigis, *A noncommutative noncocommutative Hopf algebra in “nature”*, *J. Algebra* **70** (1981), 356–374.
- [22] Y. Pei, N. Hu and M. Rosso, *Multiparameter quantum groups and quantum shuffles, (I)*, in *Quantum Affine Algebras, Extended Affine Lie Algebras, and Their Applications*, *Contemp. Math.*, **506**, Amer. Math. Soc., 2010, pp. 145–171. arXiv:0811.0129.

- [23] M.E. Sweedler, *Hopf Algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.
- [24] R. Taillefer, *Cyclic Homology of Hopf Algebras*, *K-Theory* **24** (2001), 69–85.
- [25] M. Takeuchi, $Ext_{ad}(SpR, \nu^A) \cong Br(A/k)$, *J. Algebra* **67** (1980), 436–475.
- [26] M. Takeuchi, *Matched pairs of groups and bismash products of Hopf algebras*, *Comm. Algebra* **9** (1981), 841–882.
- [27] A. van Daele and S. van Keer, *The Yang-Baxter and pentagon equation*, *Compositio Math.* **91** (1994), 201–221.
- [28] C. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, **38**, Cambridge University Press, Cambridge, 1994.

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